Stability of Hamiltonian relative equilibria in symmetric magnetically confined rigid bodies

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Dedicated to the memory of Jerrold E. Marsden

Abstract

This work studies the symmetries, the associated momentum map, and relative equilibria of a mechanical system consisting of a small axisymmetric magnetic body-dipole in an also axisymmetric external magnetic field that additionally exhibits a mirror symmetry; we call this system the "orbitron". We study the nonlinear stability of a branch of equatorial relative equilibria using the energy-momentum method and we provide sufficient conditions for their $\mathbb{T}^2$–stability that complete partial stability relations already existing in the literature. These stability prescriptions are explicitly written down in terms of some of the field parameters, which can be used in the design of stable solutions. We propose new linear methods to determine instability regions in the context of relative equilibria that allow us to conclude the sharpness of some of the nonlinear stability conditions obtained.

Key Words: Hamiltonian systems with symmetry, momentum maps, relative equilibrium, magnetic systems, orbitron, generalized orbitron, nonlinear stability/instability.

1 Introduction

Many physical systems exhibit symmetries. A number of techniques have been developed during the last two centuries to take advantage of the conservation laws that are usually associated to these invariance properties to simplify or *reduce* those systems in order to make easier the computation of their solutions. The presence of symmetries also creates natural dynamical features that generalize distinguished solutions of their non-symmetric counterparts like the so called *relative equilibria* or *relative periodic orbits*; relative equilibria are solutions of a symmetric system that coincide with one-parameter group orbits of the action that leaves that system invariant. The justification of this denomination lies in the fact that relative equilibria are equilibria for the reduced Hamiltonian system [MW74] constructed with the momentum map associated to the action, provided that this object exists. Regarding the stability of these solutions, the degeneracies caused by the presence of symmetries in a system cause drift phenomena that make non-evident the selection of a stability definition. A very reasonable choice is the concept of stability relative to a subgroup introduced in [Pat92, Pat95b] for which a number

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of energy-momentum based sufficient conditions have been formulated in the literature under different assumptions and levels of generality on the group actions involved and the momentum values at which the relative equilibrium in question takes place [Pat92, Pat95a, Pat95b, MRS88, Mon97b, Mon97a, MR99, OR99c, LS98, OR99a, OR99b, Ort98, RWL02, PRW04, OR04, MRO11]. These methods have been used, for example, in the study of the stability of relative equilibria present in different configurations of rigid bodies [LRSM92, Lew98, Pat95b], Riemann ellipsoids [FL01, ROSD08], underwater vehicles [Leo97, LM97], vortices [PM98, LP02, LMR01, LPMR11], and molecules [MR99].

In this paper we use these methods to establish sufficient conditions for the stability of various branches of relative equilibria present in a mechanical system consisting of a small axisymmetric magnetic body-dipole in an also axisymmetric external magnetic field that additionally exhibits a mirror symmetry. When the external field is created by two magnetic poles modeled by two distant "charges" [Smy39, Koz81] we call this system the ***standard orbitron***; the setup involving arbitrary external fields exhibiting the above mentioned symmetries will be referred to as the ***generalized orbitron***. The generic term ***orbitron*** will refer simultaneously to both the standard and the generalized orbitrons. This problem has been studied for a long time already: the model was introduced in the 1970s and the first theoretical and experimental results were presented in [Koz74, Koz81]. A first stability study of the regular branch of the equatorial relative equilibria of the standard orbitron is carried out in [Zub14]. These works focus mainly on the physical description of the system and do not contain a geometrically rigorous formulation of its phase space and symmetries. In the following pages we show how in this case the methods of geometric mechanics and symmetry-based stability analysis are capable of handling the study of singular relative equilibria both for the standard and the generalized orbitron and provide a precise interpretation of the obtained stability results (stability modulo a subgroup). Additionally, we complete the stability analysis by introducing new linear methods to assess the sharpness of the stability conditions.

Geometric mechanical methods have already been applied in the context of two systems involving spatially extended magnetic bodies, namely the *levitron* and the *magnetic dumbbells*. The levitron [Har83] is a magnetic spinning top in the presence of gravitation that can levitate in the air repelled by a base magnet. The stability of this dynamical phenomenon has been explored with the tools of geometric mechanics in [DE99, Dul04, KM06]. Unfortunately, in this system there are not sufficient conserved quantities available to conclude nonlinear stability using energy-momentum methods and only linear stability estimates are available. The magnetic dumbbells [Koz74] are two axisymmetric magnetic rigid bodies in space interacting contactlessly with each other; this system exhibits stable regular relative equilibria for which stability conditions have been found using the energy-momentum method in [Zub12].

The positive stability results obtained in this paper for dynamic solutions of the orbitron lead us to believe that other similar configurations that have been experimentally observed to be stable could be rigorously proved to have this property. We plan to tackle these questions with methods similar to those put at work in this paper for the orbitron in forthcoming publications.

The paper has been written using a self-contained and tutorial approach. Its structure is organized as follows: in Section 2 we present the Hamiltonian description of the orbitron by including a detailed geometric description of its phase space, equations of motion, symmetries, and associated momentum map. Section 3 contains a characterization of the relative equilibria of the orbitron that is obtained out of the critical points of the augmented Hamiltonian, constructed using the momentum map associated to the toral symmetry of this system spelled out in the preceding section. Section 4 is dedicated to the stability analysis of two branches of equatorial relative equilibria introduced in Section 3. One of these branches is singular, in the sense that it exhibits nontrivial isotropy group, and the other one is regular. The stability study is carried out for both the standard and the generalized orbitrons using the energy–momentum method, which yields in this case a set of conditions whose joint satisfaction is sufficient for the toral stability of the regular relative equilibria. Concerning the singular relative equilibria, none of these solutions can be proved to be stable using the energy–momentum method for the standard orbitron, while in the generalized case we are able to specify sufficient conditions involving both the design parameters of the external magnetic field and the dynamical features of the system that guarantee its nonlinear stability. In the second part of Section 4 we introduce new linear methods to assess the sharpness of the stability conditions; more specifically, we show that the spectral instability of a natural linearized Hamiltonian vector field that can be associated to any relative equilibrium, ensures its nonlinear instability. This result is very instrumental in our setup since it allows us, for example, to prove the nonlinear instability of the singular branch of relative equilibria of the standard orbitron and the sharpness of some of the nonlinear stability conditions obtained in the regular case. In order to improve the readability of the paper, most proofs of the results in the paper and a number of technical details about the geometry of the system that are used in those proofs, have been included in appendices at the end of the paper (Section 5).

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2 The orbitron

The standard orbitron is a a small axisymmetric magnetized rigid body (for example a small permanent magnet or a current-carrying loop) with magnetic moment μ_m, in the permanent magnetic field created by two fixed magnetic poles modeled by opposite charges placed at distance h [Smy39, Koz81] in the absence of gravity (see Figure 1); in this definition the adjective "small" refers to the size of the body in comparison with the distance $2h$ between the magnetic poles. In this section we provide the Hamiltonian description of this physical system.

Phase space. The configuration space of the orbitron is the special Euclidean group in three dimensions $SE(3) = SO(3) \times \mathbb{R}^3$. The $\mathbb{R}^3$ factor of $SE(3)$ accounts for the position of the center of mass in space of the rigid body and $SO(3)$ specifies its orientation with respect to a fixed initial frame. The orbitron is a simple mechanical system in the sense that its Hamiltonian function is of the form kinetic plus potential energy and that its phase space is the cotangent bundle $T^*SE(3)$ of its configuration space $SE(3)$ endowed with the canonical symplectic structure ω obtained as minus the differential of the corresponding Liouville one form.

As the cotangent bundle of any Lie group, $T^*SE(3)$ can be right or left trivialized in order to obtain the so called space or body coordinates, respectively (see Appendix 5.1), of the phase space. These trivializations provide an identification of the bundle $T^*SE(3)$ with the product $SE(3) \times \mathfrak{se}(3)^*$, where the symbol $\mathfrak{se}(3)^*$ stands for the dual of the Lie algebra $\mathfrak{se}(3)$ of $SE(3)$.

In this paper we will work in body coordinates unless it is specified otherwise. Using this representation, we denote by $(A, \mathbf{x})$ the elements of $SE(3) = SO(3) \times \mathbb{R}^3$ and by $((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p}))$ those of $T^*SE(3) \simeq SE(3) \times \mathfrak{se}(3)^*$ using body coordinates. The momenta $(\mathbf{\Pi}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ associated to $(A, \mathbf{x}) \in SO(3) \times \mathbb{R}^3$ are the angular and linear momentum, respectively

Equations of motion. The Hamiltonian of the orbitron is given by the sum of the kinetic $T(\mathbf{\Pi}, \mathbf{p})$ and the potential $V(A, \mathbf{x})$ energy, that is,

$$H((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p})) = T(\mathbf{\Pi}, \mathbf{p}) + V(A, \mathbf{x}) . \tag{2.1}$$

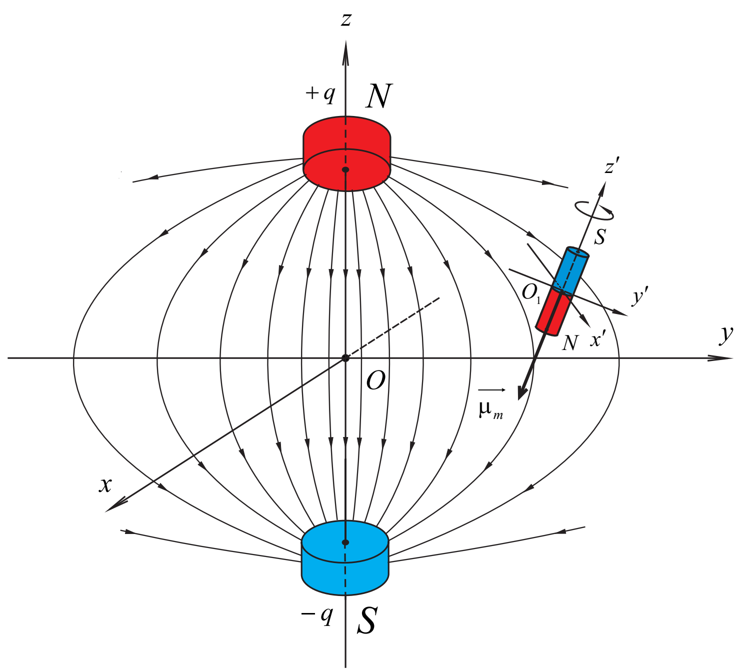

Figure 1: Schematic representation of the standard orbitron. The magnetic rigid body interacts exclusively with the fixed magnetic poles represented in the picture. The opposite poles of each fixed magnet are assumed to be very distant in comparison with the dimensions of the small rigid body; therefore, their influence is negligible and they are hence not represented.

The expression of the kinetic energy is:

$$T(\mathbf{\Pi}, \mathbf{p}) := \frac{1}{2}\mathbf{\Pi}^T \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} + \frac{1}{2M}\|\mathbf{p}\|^2, \tag{2.2}$$

where M is the mass of the axisymmetric magnetic body and $\mathbb{I}_{\text{ref}}$ the reference inertia tensor $\mathbb{I}_{\text{ref}} = \text{diag}(I_1, I_1, I_3)$. The coincidence between the first two principal moments of inertia is related to an axial symmetry with respect to the third coordinate that we assume in the body. The potential energy is given by

$$V(A, \mathbf{x}) := -\mu_m \langle \mathbf{B}(\mathbf{x}), A\mathbf{e}_3 \rangle, \tag{2.3}$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\mu_m \in \mathbb{R}$ is the magnetic moment of the axisymmetric rigid body/dipole, and $\mathbf{B}(\mathbf{x})$ is the strength of the magnetic field created by two magnetic poles/"charges" $\pm q$ placed at the points $(0, 0, h)$ and $(0, 0, -h)$, $h > 0$, that is [Koz81],

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 q}{4\pi}\left(\frac{x}{D(\mathbf{x})_+^{3/2}} - \frac{x}{D(\mathbf{x})_-^{3/2}}, \frac{y}{D(\mathbf{x})_+^{3/2}} - \frac{y}{D(\mathbf{x})_-^{3/2}}, \frac{z-h}{D(\mathbf{x})_+^{3/2}} - \frac{z+h}{D(\mathbf{x})_-^{3/2}}\right), \tag{2.4}$$

with $D(\mathbf{x})_+ = x^2 + y^2 + (z-h)^2$, $D(\mathbf{x})_- = x^2 + y^2 + (z+h)^2$, and μ_0 the magnetic permeability of vacuum. A small axisymmetric magnetized rigid body subjected to a external magnetic field of the form specified in (2.4) will be called a ***standard orbitron***.

As we will see later on, most of the results that we present in this paper hold for systems with external magnetic fields that share the following symmetry properties presented by $\mathbf{B}$ in (2.4), namely:

(i) Equivariance with respect to rotations $R^Z_{\theta_S}$ around the OZ axis:

$$\mathbf{B}(R^Z_{\theta_S}\mathbf{x}) = R^Z_{\theta_S}\mathbf{B}(\mathbf{x}) \ \text{ for } \ \theta_S \in \mathbb{R}. \tag{2.5}$$

(ii) Behavior with respect to the mirror transformation

$$(x, y, z) \longmapsto (x, y, -z) \tag{2.6}$$

according to the prescription

$$B_x(x, y, z) = -B_x(x, y, -z), \tag{2.7}$$
$$B_y(x, y, z) = -B_y(x, y, -z), \tag{2.8}$$
$$B_z(x, y, z) = B_z(x, y, -z). \tag{2.9}$$

Consider an arbitrary magnetic field in the magnetostatic approximation in a domain free of other magnetic sources that satisfies these symmetry properties. A small axisymmetric magnetized rigid body subjected to the influence of such an external field will be called a ***generalized orbitron***. The generic term ***orbitron*** will refer simultaneously to both the standard and the generalized orbitrons.

The equations of motion of the orbitron are determined by Hamilton's equations:

$$\mathbf{i}_{X_H}\omega = \mathbf{d}H, \tag{2.10}$$

where $\mathbf{i}$ denotes the interior derivative, $\mathbf{d}$ is Cartan's exterior derivative, and $X_H \in \mathfrak{X}(T^*SE(3))$ the Hamiltonian vector field associated to $H \in C^\infty(T^*SE(3))$. It can be proved (see Appendix 5.2) that in body coordinates, Hamilton's equations (2.10) amount to the set of differential equations:

$$\dot{A} = A\widehat{\mathbb{I}^{-1}_{\text{ref}}\mathbf{\Pi}}, \tag{2.11}$$
$$\dot{\mathbf{x}} = \frac{1}{M}A\mathbf{p}, \tag{2.12}$$
$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}^{-1}_{\text{ref}}\mathbf{\Pi} + A^{-1}\mathbf{B}(\mathbf{x}) \times \mathbf{e}_3, \tag{2.13}$$
$$\dot{\mathbf{p}} = \mathbf{p} \times \mathbb{I}^{-1}_{\text{ref}}\mathbf{\Pi} + \mu_m A^{-1} D\mathbf{B}(\mathbf{x})^T A\mathbf{e}_3. \tag{2.14}$$

The symbol $\widehat{\mathbb{I}^{-1}_{\text{ref}}\mathbf{\Pi}}$ stands for the antisymmetric matrix associated to the vector $\mathbb{I}^{-1}_{\text{ref}}\mathbf{\Pi} \in \mathbb{R}^3$ via the Lie algebra isomorphism $\widehat{\ }: (\mathbb{R}^3, \times) \longrightarrow (\mathfrak{so}(3), [\cdot,\cdot])$ introduced in Appendix 5.1 and D for the differential.

Toral symmetry of the orbitron and associated momentum map. The axial symmetry of the magnetic rigid body and the rotational spatial symmetry of the magnetic field created by the two poles with respect to rotations around the OZ axis endow this system with a toral symmetry which is obtained as the cotangent lift of the following action on the configuration space:

$$\begin{array}{rccc} \Phi: & (\mathbb{T}^2 = S^1 \times S^1) \times SE(3) & \longrightarrow & SE(3) \\ & \left(\left(e^{i\theta_S}, e^{i\theta_B}\right), (A, \mathbf{x})\right) & \longmapsto & (R^Z_{\theta_S} A R^Z_{-\theta_B}, R^Z_{\theta_S}\mathbf{x}), \end{array} \tag{2.15}$$

where R^Z_θ denotes the rotation matrix around the third axis by an angle θ. The first circle action involving $R^Z_{\theta_S}$ implies a spatial rotation of the center of mass of the body and the second one, given by $R^Z_{\theta_B}$, accounts for a rotation of the magnetic body around its symmetry axis. In Appendix 5.3 we show that the cotangent lift, also denoted by Φ, is a canonical symmetry given by

$$\begin{array}{rccc} \Phi: & (\mathbb{T}^2 = S^1 \times S^1) \times T^*SE(3) & \longrightarrow & T^*SE(3) \\ & \left(\left(e^{i\theta_S}, e^{i\theta_B}\right), ((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p}))\right) & \longmapsto & ((R_{\theta_S} A R_{-\theta_B}, R_{\theta_S}\mathbf{x}), (R_{\theta_B}\mathbf{\Pi}, R_{\theta_B}\mathbf{p}))\,. \end{array} \tag{2.16}$$

This has an invariant momentum map associated $\boldsymbol{J} : T^*SE(3) \longrightarrow \mathbb{R}^2$:

$$\boldsymbol{J}\left((A, \mathbf{x}), (\boldsymbol{\Pi}, \mathbf{p})\right) = \left(\langle A\boldsymbol{\Pi} + \mathbf{x} \times A\mathbf{p}, \mathbf{e}_3\rangle, -\langle \boldsymbol{\Pi}, \mathbf{e}_3\rangle\right). \tag{2.17}$$

A straightforward computation shows that the Hamiltonian of the orbitron is invariant with respect to the action (2.16), that is,

$$H \circ \Phi_{\left(e^{i\theta_S}, e^{i\theta_B}\right)} = H, \quad \text{for any} \quad \left(e^{i\theta_S}, e^{i\theta_B}\right) \in \mathbb{T}^2,$$

which, by Noether's Theorem [AM78, Theorem 4.2.2], allows us to conclude that the level sets of the momentum map (2.17) are preserved by the associated Hamiltonian dynamics, that is, if F_t is the flow of the vector field X_H then $\mathbf{J} \circ F_t = \mathbf{J}$ for any t.

The action (2.16) has two isotropy subgroups, namely, the identity $\{e\}$ and the diagonal circle $K := \left\{\left(e^{i\theta}, e^{i\theta}\right) \mid e^{i\theta} \in S^1\right\}$. The orbit type submanifold $(T^*SE(3))_H$ is given by

$$(T^*SE(3))_K = \left\{\left(\left(R^Z_\theta, a\mathbf{e}_3\right), (b\mathbf{e}_3, c\mathbf{e}_3)\right) \mid \theta, a, b, c \in \mathbb{R}\right\}, \tag{2.18}$$

and $(T^*SE(3))_{\{e\}} = T^*SE(3) \setminus (T^*SE(3))_K$. The bifurcation lemma (see for instance [OR04, Proposition 4.5.12]) guarantees that the restriction of the momentum map to the regular isotropy type $T^*SE(3)_{\{e\}}$ is a submersion and that it has rank one at points in the isotropy type $(T^*SE(3))_K$.

3 Relative equilibria of the orbitron

In this section we specify the equations that characterize the relative equilibria of the orbitron with respect to its toral symmetry.

Relative equilibria: setup and background. Consider a vector field $X \in \mathfrak{X}(M)$ on a manifold M that is equivariant with respect to action of a Lie group G on it. We say that the point $m \in M$ is a relative equilibrium with velocity $\xi \in \mathfrak{g}$ if the value of vector field at that point coincides with the infinitesimal generator ξ_M associated to ξ, that is,

$$X(m) = \xi_M(m). \tag{3.1}$$

The Lie algebra element $\xi \in \mathfrak{g}$ is called the velocity of the relative equilibrium. This defining property is equivalent to saying that the flow F_t associated to the vector field X at the point $m \in M$ coincides with the one-parameter Lie subgroup of G generated by $\xi \in \mathfrak{g}$, that is,

$$F_t(m) = \exp t\xi \cdot m, \tag{3.2}$$

where exp is the Lie group exponential map $\exp : \mathfrak{g} \to G$. In the Hamiltonian setup, relative equilibria have a very convenient characterization that uses the critical points of a function instead of the equilibria of the vector field $X - \xi_M$, as in (3.1). Indeed, consider now a symmetric Hamiltonian system $(M, \omega, H, G, \mathbf{J} : M \to \mathfrak{g}^*)$ and assume that the momentum map $\mathbf{J}$ is coadjoint equivariant; it can be shown [AM78] that the point $m \in M$ is a relative equilibrium of the Hamiltonian vector field X_H with velocity $\xi \in \mathfrak{g}$ if and only if

$$\mathbf{d}\left(H - \mathbf{J}^\xi\right)(m) = 0, \tag{3.3}$$

where $\mathbf{J}^\xi := \langle \mathbf{J}, \xi\rangle$. The combination $H - \mathbf{J}^\xi$ is usually referred to as the ***augmented Hamiltonian***. If the relative equilibrium $m \in M$ is such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$ and we denote its isotropy subgroup with respect to the G action by G_m, the law of conservation of the isotropy [OR04] and Noether's Theorem imply [OR99c, Theorem 2.8] that $\xi \in \operatorname{Lie}\left(N_{G_\mu}(G_m)\right)$, where G_μ is the coadjoint isotropy of

$\mu \in \mathfrak{g}^*$ and $N_{G_\mu}(G_m)$ is the normalizer group of G_m in G_μ (note that $G_m \subset G_\mu$ necessarily due to the equivariance of the momentum map). Finally, notice that the velocity of a relative equilibrium with nontrivial isotropy is not uniquely defined; indeed, it is clear in (3.2) that if $\xi \in \mathfrak{g}$ is a velocity for the relative equilibrium m, then so is $\xi + \eta$ for any $\eta \in \mathrm{Lie}\,(G_m)$.

Relative equilibria equations of the orbitron. The next proposition, proved in Appendix 5.4, specifies the critical point equations (3.3) in the case of the orbitron and shows the existence of branches of relative equilibria whose stability we will study in the next section.

Proposition 3.1 *Consider the orbitron system introduced in Section 2 whose Hamiltonian function is given by (2.1) and let* $\mathbf{z} = ((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p})) \in T^*SE(3)$. *Then:*

(i) *The point* $\mathbf{z}$ *is a relative equilibrium of the orbitron with velocity* $(\xi_1, \xi_2) \in \mathbb{R}^2$ *with respect to the toral symmetry introduced in Section 2 if and only if the following identities are satisfied:*

$$\mu_m \left[\mathbf{B}(\mathbf{x}) \times A\mathbf{e}_3\right] + \xi_1 \left[A\mathbf{p} \times (\mathbf{x} \times \mathbf{e}_3) - A\mathbf{\Pi} \times \mathbf{e}_3\right] = 0, \tag{3.4}$$

$$-\mu_m D\mathbf{B}(\mathbf{x})^T (A\mathbf{e}_3) - \xi_1 \left(A\mathbf{p} \times \mathbf{e}_3\right) = 0, \tag{3.5}$$

$$\mathbb{I}_{\mathrm{ref}}^{-1}\mathbf{\Pi} + \xi_2 \mathbf{e}_3 - \xi_1 A^{-1}\mathbf{e}_3 = 0, \tag{3.6}$$

$$\frac{1}{M}\mathbf{p} - \xi_1 A^{-1}\left(\mathbf{e}_3 \times \mathbf{x}\right) = 0. \tag{3.7}$$

(ii) *Consider now* $A_0 = R^Z_{\theta_0}$, $\mathbf{x}_0 = (x, y, 0)$, $\mathbf{\Pi}_0 = I_3\left(\xi_1 - \xi_2\right)\mathbf{e}_3$ *and* $\mathbf{p}_0 = M\xi_1 A_0^{-1}(-y, x, 0)$. *The point* $\mathbf{z}_0 = ((A_0, \mathbf{x}_0), (\mathbf{\Pi}_0, \mathbf{p}_0))$ *is a relative equilibrium of the standard orbitron with velocity* (ξ_1, ξ_2), *where* ξ_2 *is an arbitrary real number and* ξ_1 *is either arbitrary when* $\mathbf{x}_0 = \mathbf{0}$ *or*

$$\xi_1 = \pm\left(-\frac{3h\mu_m q\mu_0}{2\pi M D(\mathbf{x}_0)^{5/2}}\right)^{1/2}, \tag{3.8}$$

when $\mathbf{x}_0 \neq \mathbf{0}$. *In view of the expression (3.8) for the spatial velocity* ξ_1, *the existence of the latter relative equilibrium is only guaranteed when* $\mu_m q < 0$.

(iii) *The conclusions in the previous part also hold for the generalized orbitron. In this situation* $B_z(x, y, z) = f(x^2 + y^2, z)$ *for some* $f \in C^\infty(\mathbb{R}^2)$, *and the spatial velocity* ξ_1 *of the relative equilibria with* $\mathbf{x}_0 \neq \mathbf{0}$ *is given by*

$$\xi_1 = \pm\left(-\frac{2}{M}\mu_m f_1'\right)^{1/2}, \tag{3.9}$$

where $f_1' = \left.\dfrac{\partial f(v,z)}{\partial v}\right|_{v=x^2+y^2, z=0}$. *In view of the expression of the spatial velocity* ξ_1 *in (3.9), the existence of this relative equilibrium is only guaranteed when* $\mu_m f_1' < 0$.

The relative equilibria in these statements for which $\mathbf{x}_0 \neq \mathbf{0}$ *(respectively* $\mathbf{x}_0 = \mathbf{0}$*) have trivial (respectively nontrivial* K*) isotropy and hence belong to the orbit type* $\left(T^*SE(3)\right)_{\{e\}}$ *(respectively* $\left(T^*SE(3)\right)_K$*); we will refer to them as* ***regular relative equilibria*** *(respectively* ***singular relative equilibria****).*

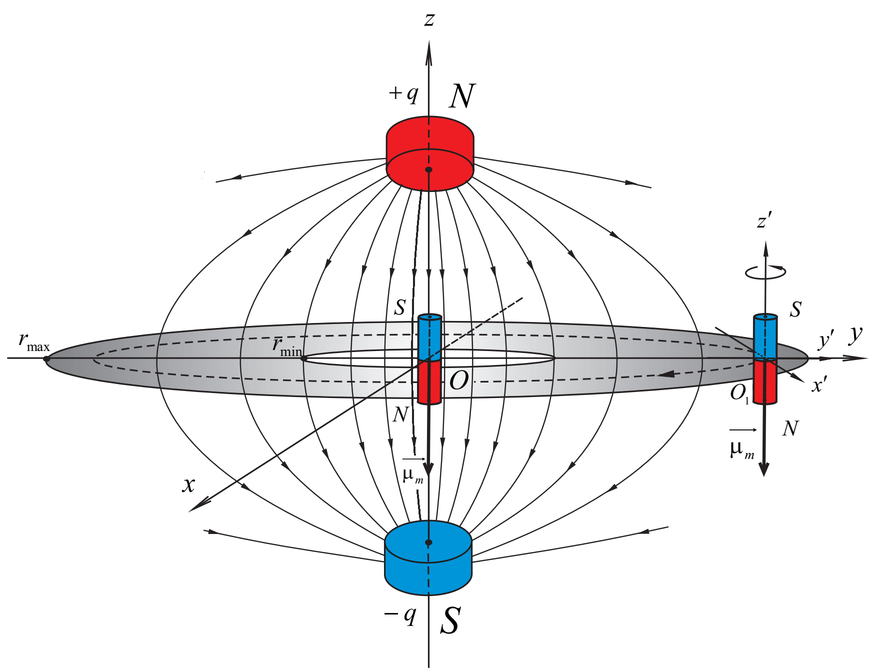

Figure 2: Regular (dipole in the right side of the picture) and singular (dipole at the origin) relative equilibria of the standard orbitron. The symbols r_{min} and r_{max} represent the stability region in configuration space determined by the stability conditions in the expression (4.2). Theorem 4.5 shows that dipoles in a regular relative equilibrium configuration that rotate around the origin with a radius in the interval $[r_{min}, r_{max}]$ and with a body angular velocity that satisfies (4.3) are stable. The singular relative equilibria are always unstable (see Theorem 4.14).

4 Stability analysis of the relative equilibria of the orbitron

In this section we study the stability properties of the branches of relative equilibria of the orbitron introduced in the second and third parts of Proposition 3.1.

The energy–momentum method. As we already explained in the introduction, the degeneracies present in symmetric systems cause various drift phenomena that complicate the selection of a stability criterion. The most natural and fruitful choice is that of stability relative to a subgroup, introduced in [Pat92] for relative equilibria and in [OR99a] for relative periodic orbits.

Definition 4.1 *Let $X \in \mathfrak{X}(M)$ be a G–equivariant vector field on the G–manifold M and let G' be a subgroup of G. A relative equilibrium $m \in M$ of X, is called G'–**stable**, or **stable modulo** G', if for any G'–invariant open neighborhood V of the orbit $G' \cdot m$, there is an open neighborhood $U \subset V$ of m, such that if F_t is the flow of the vector field X and $u \in U$, then $F_t(u) \in V$ for all $t \geq 0$.*

In the Hamiltonian setup there exists a variety of Dirichlet type results that provide sufficient conditions for the G_μ–stability of a given relative equilibrium, where $\mu \in \mathfrak{g}^*$ is the momentum value in which it is sitting and G_μ is its coadjoint isotropy. The reason why the subgroup G_μ arises naturally is clear if we look at the stability problem from the symplectic reduction point of view; more explicitly, consider a symmetric Hamiltonian system $(M, \omega, H, G, \mathbf{J} : M \to \mathfrak{g}^*)$ that exhibits a relative equilibrium at the point $m \in M$ such that $\mathbf{J}(m) = \mu$. Suppose that the momentum map $\mathbf{J}$ is coadjoint equivariant and that the coadjoint isotropy G_μ acts freely and properly on the momentum fiber $\mathbf{J}^{-1}(\mu)$; in these conditions, the quotient space $\mathbf{J}^{-1}(\mu)/G_\mu$ is naturally symplectic [Mey73, MW74] and the G-equivariant

Hamiltonian vector field associated to H projects onto another Hamiltonian vector field in which the relative equilibrium m becomes a standard equilibrium. The importance of this construction in our context comes from the fact that the standard Lyapunov stability of the reduced equilibrium is equivalent to the G_μ–stability of the relative equilibrium.

The following result, known as the ***energy–momentum method***, provides a sufficient condition for the G_μ–stability of a given relative equilibrium. This result has been introduced at different levels of generality in [Pat92, OR99c, PRW04, MRO11].

Theorem 4.2 (Energy-momentum method) *Let (M, ω, H) be a symplectic Hamiltonian system with a symmetry given by the Lie group G acting properly on M with an associated coadjoint equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $m \in M$ be a relative equilibrium such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$ and assume that the coadjoint isotropy subgroup G_μ is compact. Let $\xi \in \mathrm{Lie}N_{G_\mu}(G_m)$ be a velocity of the relative equilibrium. If the quadratic form*

$$\mathbf{d}^2(H - \mathbf{J}^\xi)(m)|_{W\times W} \tag{4.1}$$

is definite for some (and hence for any) subspace W such that

$$\ker T_m\mathbf{J} = W \oplus T_m(G_\mu \cdot m),$$

*then m is a G_μ–stable relative equilibrium. If $\dim W = 0$, then m is always a G_μ–stable relative equilibrium. The quadratic form $\mathbf{d}^2(H - \mathbf{J}^\xi)(m)|_{W\times W}$, will be called the **stability form** of the relative equilibrium m and W a **stability space**.*

Remark 4.3 Even though we work exclusively in the Hamiltonian setup, this criterion has elaborate counterparts in the Lagrangian side [SLM91, RO06].

Remark 4.4 The statement in Theorem 4.2 can be generalized to the context of Hamiltonian actions on Poisson manifolds and can be stated so that one can take advantage of existing Casimirs or other non-symmetry related conserved quantities in order to prove the stability of a given relative equilibrium [OR99b, Theorem 4.8]. More explicitly, if in the conditions of Theorem 4.2 there exists a set of G_μ–invariant conserved quantities $C_1, \ldots, C_n : M \to \mathbb{R}$ for which

$$\mathbf{d}(H - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(H - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m)|_{W\times W}$$

is definite for some (and hence for any) W such that

$$\ker \mathbf{d}C^1(m) \cap \ldots \cap \ker \mathbf{d}C^n(m) \cap \ker T_m\mathbf{J} = W \oplus T_m(G_\mu \cdot m),$$

then m is a G_μ–stable relative equilibrium.

Nonlinear stability of the orbitron relative equilibria. The application of the energy-momentum method to the relative equilibria of the orbitron introduced in Proposition 3.1 makes possible the determination of sizeable regions in parameter space for which those solutions are G_μ-stable ($\mathbb{T}^2$-stable in this case). We spell this out in the statement of the following theorem whose proof is provided in the Appendix 5.5.

Theorem 4.5 *Consider the relative equilibria introduced in Proposition 3.1. Then:*

(i) *The regular relative equilibria of the standard orbitron in part* **(ii)** *of Proposition 3.1, that is, those for which $\mathbf{x}_0 \neq \mathbf{0}$, are $\mathbb{T}^2$–stable whenever the following three inequalities are satisfied:*

$$\frac{2}{3} < \frac{r^2}{h^2} < 4, \tag{4.2}$$

$$\operatorname{sign}(\xi_1^0) I_3 \xi_2 < - \mid \xi_1^0 \mid \left(I_1 - I_3 + \frac{2}{3} M \frac{(r^2+h^2)h^2}{3r^2 - 2h^2} \right), \tag{4.3}$$

where $r^2 = \|\mathbf{x}_0\|^2$, $\xi_1^0 = \pm \left(-\frac{3h\mu_m q \mu_0}{2\pi M D(\mathbf{x}_0)^{5/2}} \right)^{1/2}$, and $\mu_m q < 0$.

The singular relative equilibria ($\mathbf{x}_0 = \mathbf{0}$) are always formally unstable, in the sense that the stability form (4.1) exhibits a nontrivial signature.

(ii) *The regular relative equilibria of the generalized orbitron in part* **(iii)** *of Proposition 3.1 are $\mathbb{T}^2$–stable whenever the following conditions hold:*

$$\mu_m f_1' < 0, \tag{4.4}$$

$$\mu_m \left(2f_1' + r^2 f_1'' \right) < 0, \tag{4.5}$$

$$\mu_m f_2'' < 0, \tag{4.6}$$

$$\operatorname{sign}(\xi_1^0) I_3 \xi_2 < -|\xi_1^0| \left((I_1 - I_3) + \frac{1}{2} M \left(\frac{f_0}{f_1'} + 4r^2 \frac{f_1'}{f_2''} \right) \right), \tag{4.7}$$

where $r^2 = \|\mathbf{x}_0\|^2$, $f \in C^\infty(\mathbb{R}^2)$ is the function such that $B_z(x,y,z) = f(r^2, z)$, $f_0 = f(r^2, 0)$, $f_1' = \left.\frac{\partial f(v,z)}{\partial v}\right|_{v=r^2, z=0}$, $f_1'' = \left.\frac{\partial^2 f(v,z)}{\partial v^2}\right|_{v=r^2, z=0}$, $f_2'' = \left.\frac{\partial^2 f(v,z)}{\partial z^2}\right|_{v=r^2, z=0}$, and $\xi_1^0 = \pm\left(-\frac{2}{M}\mu_m f_1'\right)^{1/2}$.

The singular branch ($\mathbf{x}_0 = \mathbf{0}$) is $\mathbb{T}^2$–stable if the following conditions are satisfied:

$$\mu_m f_1' < 0, \tag{4.8}$$

$$\mu_m f_2'' < 0, \tag{4.9}$$

$$\xi_1^2 < -\frac{2}{M}\mu_m f_1', \tag{4.10}$$

$$\operatorname{sign}(\xi_1)\Pi_0 > \frac{I_1 \xi_1^2 - \mu_m f_0}{|\xi_1|}, \tag{4.11}$$

where $\Pi_0 = I_3(\xi_1 - \xi_2)$ and we use the same notation as above for f_0, f_1', and f_2'', replacing $v = r^2$ by $v = 0$. When $\mu_m f_0 < 0$ and $\frac{f_0}{f_1'} < \frac{2}{M} I_1$, the conditions (4.10) and (4.11) can be replaced by the following single ξ_1–independent optimal condition:

$$|\Pi_0| > 2\sqrt{-\mu_m f_0 I_1}. \tag{4.12}$$

This optimal condition is achieved by using the spatial velocities $\xi_1 = \pm\left(-\frac{1}{I_1}\mu_m f_0\right)^{1/2}$; the positive (respectively negative) sign for the velocity corresponds to positive (respectively negative) values of Π_0.

Remark 4.6 The right inequality in (4.2) appears already in Kozorez's [Koz81] study of the standard orbitron but it does not ensure by itself the nonlinear stability of this symmetric configuration. We will refer to this inequality as the ***Kozorez condition***. The extension of this inequality in the context of the generalized orbitron is given by (4.5). The stability conditions for the regular branch of relative equilibria of the standard orbitron have also been obtained in [Zub14].

Remark 4.7 The formal instability of the singular branch of the standard orbitron is not informative about its actual nonlinear stability or instability. This point is determined via a complementary spectral stability analysis of the linearized system that we carry out later on in Theorem 4.14 and that allows us to conclude the nonlinear instability of this singular branch of relative equilibria.

Remark 4.8 The proof of the theorem presented in Appendix 5.5 consists of studying the definiteness of the stability form (4.1) introduced in Theorem 4.2. A quick dimension count shows that the stability spaces corresponding to the regular and singular branches of relative equilibria are eight and ten dimensional, respectively. The need of determining the sign of the eigenvalues of stability forms in high dimensions like ours has motivated the introduction in the literature of various block diagonalizations for it based on arguments of dynamic [SLM91, RO06] or kinematic [OR99c] nature. An elementary but important observation that we point out in the proof of this theorem is that in order to ensure the stability of the relative equilibrium in question there is no need to compute the eigenvalues of the stability form but only to determine its signature; the relevance of this statement lies in the fact that by Sylvester's Law of Inertia, the signature is invariant by conjugation with respect to invertible matrices and hence can be read out of the pivots of the matrix obtained by performing Gaussian elimination on the stability form. Unlike the situation faced when computing eigenvalues, Gaussian elimination can be carried out formally and not just numerically in virtually any dimension. This remark is of much importance for non-simple mechanical systems for which dynamic block diagonalizations similar to those cited above are rarely available.

Remark 4.9 Conditions (4.8)–(4.11) can be used in the design of magnetic fields capable of confining magnetic rigid bodies that exhibit exclusively body rotation. This is the working principle of devices such as magnetic contactless flywheels or levitrons. In the case of flywheels, up until now only actively controlled versions have been developed; as to the levitron, the potentials that have been considered so far [DE99, Dul04, KM06] do not allow to conclude nonlinear stability using the methods put at work in Theorem 4.5 and only the spectral stability of the corresponding linearized systems has been considered. We plan to explore in detail these systems in a future publication.

Remark 4.10 The stability conditions (4.8)–(4.11) are also valid for magnetic fields in the presence of other sources like currents or time varying electric fields, as long as the following two conditions are satisfied:

$$\frac{\partial B_x(\mathbf{0})}{\partial z} = \frac{\partial B_y(\mathbf{0})}{\partial z} = 0. \tag{4.13}$$

This statement, shown at the end of proof of Theorem 4.5 in the appendix, is important because the stability conditions (4.8)–(4.11) cannot hold for very common magnetic fields without additional sources like, for instance, magnetic fields obtained as the gradient of symmetric potentials (for example those generated by permanent magnets). Indeed, suppose that the magnetic field is of the form $\mathbf{B}(\mathbf{x}) = -\nabla U(\mathbf{x})$, where $\mathbf{x} = (x, y, z)$ and U is an invariant function $U(\mathbf{x}) = F(x^2 + y^2, z)$. This field automatically satisfies the Ampère-Maxwell law in the absence of currents and external electric fields, namely, $\nabla \times \mathbf{B} = 0$; at the same time, the Gauss-Maxwell law requires that $\nabla \cdot \mathbf{B} = 0$, which amounts to

$$4F_1(x^2 + y^2, z) + 4(x^2 + y^2)F_{11}(x^2 + y^2, z) + F_{22}(x^2 + y^2, z) = 0,$$

where F_1 and F_{11} (respectively F_{22}) are the first and the second derivatives with respect to the first (respectively second) argument of F. This equality implies relations between the higher order derivatives of F, that is, if we take derivatives with respect to the first and the second arguments we obtain:

$$8F_{11}(x^2+y^2,z)+4(x^2+y^2)F_{111}(x^2+y^2,z)+F_{221}(x^2+y^2,z)=0,$$
$$4F_{12}(x^2+y^2,z)+4(x^2+y^2)F_{112}(x^2+y^2,z)+F_{222}(x^2+y^2,z)=0.$$

For the singular relative equilibrium the second equality reduces to

$$4F_{12}(0,0)+F_{222}(0,0)=0. \tag{4.14}$$

At the same time, it is easy to check the stability conditions (4.8) and (4.9) require that $F_{12}(0,0)$ and $F_{222}(0,0)$ are non-zero and have the same sign, which is clearly incompatible with (4.14).

Linear stability and instability analysis tools for relative equilibria. The use of the energy-momentum method provides sufficient but not necessary nonlinear stability conditions. More specifically, there is no guarantee that the stability regions determined by the inequalities in the statement of Theorem 4.5 are optimal in the sense that as soon as those conditions are violated stability disappears. In the context of stability studies for standard equilibria one usually proceeds by examining the spectral stability of the linearization at the equilibrium of the vector field in question, that is, when the sufficient stability conditions obtained via a Dirichlet type criterion are violated, one looks for eigenvalues of the linearization that exhibit a nonzero real part, whose existence would imply the nonlinear instability of the equilibrium of the original vector field.

This way to proceed can be extended in the context of regular relative equilibria by looking at the spectral stability of the linearization of the reduced Hamiltonian vector field at the equilibrium corresponding to the relative equilibrium in the symplectic Marsden–Weinstein reduced space [MW74]. Even though in the singular case, there exist reduced spaces that generalize the Marsden–Weinstein reduced space [SL91, OR06a, OR06b], the equivalence between G_μ-stability of a relative equilibrium and standard nonlinear stability of the corresponding reduced equilibrium is a more delicate issue, which makes necessary the formulation of a criterion that, as the energy-momentum method in Theorem 4.2, provides a linear stability analysis tool for relative equilibria whose formulation does not need reduction; such a statement is provided in the next proposition, whose proof can be found in the appendix, and we will apply it later on to the branches introduced in Proposition 3.1 whose nonlinear stability was studied in Theorem 4.5. In order to fix the notation and to make the presentation self contained, we start by recalling the notion of linearization of a vector field at an equilibrium point.

Definition 4.11 *Let $X \in \mathfrak{X}(M)$ be a vector field on the manifold M and let $m_0 \in M$ be an equilibrium point, that is, $X(m_0)=0$. The* ***linearization*** *X' of X at the point m_0 is a vector field $X' \in \mathfrak{X}(T_{m_0}M)$ on the vector space $T_{m_0}M$, defined by*

$$\begin{array}{rcl} X' : T_{m_0}M & \longrightarrow & T_{m_0}M \\ v & \longmapsto & \left.\frac{d}{d\lambda}\right|_{\lambda=0} T_{m_0}F_\lambda \cdot v, \end{array}$$

where F_λ is the flow of X. The eigenvalues of the linear map $X' : T_{m_0}M \longmapsto T_{m_0}M$ are called the ***characteristic exponents*** *of X at m_0.*

Proposition 4.12 *Let G be a Lie group acting canonically and properly on the symplectic manifold (M,ω) and suppose that there exists a coadjoint equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ that can be associated to it. Let $H \in C^\infty(M)^G$ be a G–invariant Hamiltonian and let $m \in M$ be a relative equilibrium of the corresponding G–equivariant Hamiltonian vector field X_H with velocity $\xi \in \mathfrak{g}$. Consider a G_m–invariant stability space W such that*

$$\ker T_m\mathbf{J} = W \oplus T_m\left(G_\mu \cdot m\right),$$

with $\mu := \mathbf{J}(m)$ *and* $G_\mu \subset G$ *the coadjoint isotropy of* $\mu \in \mathfrak{g}^*$. *Then:*

(i) (W, ω_W) *with* $\omega_W := \omega(m)|_W$ *is a symplectic vector subspace of* $(T_m M, \omega(m))$.

(ii) *There exists a symplectic slice* (S, ω_S) *at* $m \in M$ *such that* $(T_m S, \omega_S(m)) = (W, \omega_W)$.

(iii) *The Hamiltonian vector field* $X_{H_S^\xi} \in \mathfrak{X}(S)$ *in* S *associated to the Hamiltonian function* $H_S^\xi := \left(H - \mathbf{J}^\xi\right)\big|_S$ *exhibits an equilibrium at the point* $m \in S \subset M$.

(iv) *The linearization* $X'_{H_S^\xi} \in \mathfrak{X}(T_m S) = \mathfrak{X}(W)$ *of* $X_{h_S^\xi}$ *at* $m \in S$ *coincides with the linear Hamiltonian vector field* X_Q *on* (W, ω_W) *that has as Hamiltonian vector field the stability form*

$$Q(w) := \mathbf{d}^2 \left(H - \mathbf{J}^\xi\right)(m)(w, w), \qquad w \in W.$$

(v) *Suppose that the two tangent spaces* $T_m (G_\mu \cdot m)$ *and* $T_m (G \cdot m)$ *coincide. Then*

$$T_m M = W \oplus W^\omega. \tag{4.15}$$

Additionally, let $H^\xi := H - \mathbf{J}^\xi \in C^\infty(M)$ *be the augmented Hamiltonian and let* $X'_{H^\xi} \in \mathfrak{X}(T_m M)$ *be the linearization of the Hamiltonian vector field* X_{H^ξ} *at* m. *Then*

$$X_Q = \mathbb{P}_W X'_{H^\xi} \boldsymbol{i}_W, \tag{4.16}$$

where $\boldsymbol{i}_W : W \hookrightarrow T_m M$ *is the inclusion,* $\mathbb{P}_W : T_m M \longrightarrow W$ *is the projection according to* (4.15), *and* X'_{H^ξ} *is the linearization of* X_{H^ξ} *at* m.

(vi) *If the linear vector field* X_Q *is spectrally unstable in the sense that it exhibits eigenvalues with a nontrivial real part, then the relative equilibrium* $m \in M$ *of* X_H *is nonlinearly* K*–unstable, for any subgroup* $K \subset G$.

We now provide a result that spells out how to compute the linearization of a Hamiltonian vector field at an equilibrium for systems whose phase space is the cotangent bundle of a Lie group. The following proposition expresses the linearization that we need in terms of a linear map on the Euclidean vector space formed by the direct product of the Lie algebra and its dual.

Proposition 4.13 *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$ *and let* T^*G *be its cotangent bundle endowed with the canonical symplectic form. Consider now the body coordinates (left trivialized) expression* $G \times \mathfrak{g}^*$ *of* T^*G *and let* $H \in C^\infty(G \times \mathfrak{g}^*)$ *be a Hamiltonian function whose associated Hamiltonian vector field* X_H *exhibits an equilibrium at the point* $(g, \mu) \in G \times \mathfrak{g}^*$. *Then:*

(i) *Let* $\varphi : G \times (G \times \mathfrak{g}^*) \to G \times \mathfrak{g}^*$ *be the cotangent lift of the action by left translations of* G *on* G *expressed in body coordinates. Let* $H^g := H \circ \varphi_g$*; the Hamiltonian vector field* X_{H^g} *exhibits an equilibrium at the point* (e, μ).

(ii) *Let* $\Phi_g := T_{(e,\mu)}\varphi_g : T_{(e,\mu)} (G \times \mathfrak{g}^*) \simeq \mathfrak{g} \times \mathfrak{g}^* \longrightarrow T_{(g,\mu)} (G \times \mathfrak{g}^*)$ *and let* $Q \in C^\infty \left(T_{(g,\mu)} (G \times \mathfrak{g}^*)\right)$ *(respectively* $Q^g \in C^\infty(\mathfrak{g} \times \mathfrak{g}^*)$*) be the quadratic form associated to the second derivative of* H *at* (g, μ) *(respectively of* H^g *at* (e, μ)*). Then*

$$Q^g = Q \circ \Phi_g \tag{4.17}$$

and the associated linear Hamiltonian vector fields considered as linear maps satisfy:

$$\Phi_g \circ X_{Q^g} = X_Q \circ \Phi_g. \tag{4.18}$$

(iii) *The linearization* $X_{Q^g} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ *is given by:*

$$X_{Q^g}(\xi, \tau) = \left(\pi_{\mathfrak{g}^*}(\mathrm{Hess}(\xi,\tau)), -\pi_{\mathfrak{g}}(\mathrm{Hess}(\xi,\tau)) + \mathrm{ad}^*_{\pi_{\mathfrak{g}^*}\mathrm{Hess}(\xi,\tau)}\mu \right), \quad \textit{for any} \quad (\xi,\tau) \in \mathfrak{g} \times \mathfrak{g}^*, \tag{4.19}$$

where $\pi_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$, $\pi_{\mathfrak{g}^*} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ *are the canonical projections and* $\mathrm{Hess} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ *is the linear map associated to the Hessian of* H^g *at* (e,μ) *by the relation*

$$\langle \mathrm{Hess}(\xi,\tau), (\eta,\rho) \rangle = \mathbf{d}^2 H^g(e,\mu)((\xi,\tau),(\eta,\rho)), \quad (\xi,\tau),(\eta,\rho) \in \mathfrak{g} \times \mathfrak{g}^*.$$

We now implement the expression for the linearization of a Hamiltonian vector field obtained in this proposition, in the particular case of the cotangent bundle $T^*SE(3)$. Let $H \in C^\infty(T^*(SE(3)))$ be a Hamiltonian function and let X_H be the corresponding Hamiltonian vector field that we assume has an equilibrium at the point $\mathbf{z}_0 = ((A_0, \mathbf{x}_0), (\mathbf{\Pi}_0, \mathbf{p}_0))$, that is, $\mathbf{d}H(\mathbf{z}_0) = 0$. Let $g = (A_0, \mathbf{x}_0) \in SE(3)$ and let $\mathbf{z} = ((I, \mathbf{0}), (\mathbf{\Pi}_0, \mathbf{p}_0))$; using the notation in the previous proposition, it is clear that $\mathbf{z}_0 = \varphi_g(\mathbf{z})$. Let $\mathrm{Hess}(\mathbf{z}) : \mathfrak{se}(3) \times \mathfrak{se}(3)^* \to \mathfrak{se}(3) \times \mathfrak{se}(3)^*$ be the linear map associated to the Hessian of $H \circ \varphi_g$ at $\mathbf{z}$, that is, for any $\mathbf{v}, \mathbf{w} \in T_{\mathbf{z}}(T^*SE(3)) \simeq \mathfrak{se}(3) \times \mathfrak{se}(3)^*$,

$$\langle \mathbf{v}, \mathrm{Hess}(\mathbf{z})\mathbf{w} \rangle = \mathbf{d}^2(H \circ \varphi_g)(\mathbf{z})(\mathbf{v}, \mathbf{w}).$$

Now, given $\mathbf{v} = (\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}) \in \mathfrak{se}(3) \times \mathfrak{se}(3)^*$, define the projection (also available also for the $\boldsymbol{\delta}\mathbf{x}$, $\boldsymbol{\delta\Pi}$, $\boldsymbol{\delta}\mathbf{p}$ components):

$$\begin{array}{rcl} \boldsymbol{\pi}_{\delta A} : \mathfrak{se}(3) \times \mathfrak{se}(3)^* & \longrightarrow & \mathbb{R}^3 \\ (\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}) & \longmapsto & \delta A \end{array} \tag{4.20}$$

By relations (4.18) and (4.19) in Proposition 4.13, and the expression (5.10), the linearization X'_H of X_H at $\mathbf{z}_0$ is given by

$$X'_H = \Phi_g \circ X'_{H^g} \circ \Phi_{g^{-1}}, \tag{4.21}$$

where $X'_{H^g} : \mathfrak{se}(3) \times \mathfrak{se}(3)^* \simeq \mathbb{R}^{12} \to \mathfrak{se}(3) \times \mathfrak{se}(3)^* \simeq \mathbb{R}^{12}$ is the linear map determined by the twelve by twelve matrix

$$X'_{H^g} = \begin{pmatrix} \boldsymbol{\pi}_{\boldsymbol{\delta\Pi}} \mathrm{Hess}(\mathbf{z}_0) \\ \boldsymbol{\pi}_{\boldsymbol{\delta}\mathbf{p}} \mathrm{Hess}(\mathbf{z}_0) \\ -\boldsymbol{\pi}_{\delta A} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{\Pi}}_0 \boldsymbol{\pi}_{\boldsymbol{\delta\Pi}} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{p}}_0 \boldsymbol{\pi}_{\boldsymbol{\delta}\mathbf{p}} \mathrm{Hess}(\mathbf{z}_0) \\ -\boldsymbol{\pi}_{\boldsymbol{\delta}\mathbf{x}} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{p}}_0 \boldsymbol{\pi}_{\boldsymbol{\delta\Pi}} \mathrm{Hess}(\mathbf{z}_0) \end{pmatrix}. \tag{4.22}$$

This expression should be understood as a vertical concatenation of four matrices with three rows and twelve columns each. More explicitly, given that for any $\mathbf{v} = \left(\widehat{\delta A}, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}\right) \in \mathfrak{se}(3) \times \mathfrak{se}(3)^*$, $\Phi_g(\mathbf{v}) = \left(A_0\widehat{\delta A}, A_0\boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}\right) \in T_{\mathbf{z}_0}(T^*SE(3))$, we can write

$$X'_H\left(A_0\widehat{\delta A}, A_0\boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}\right) = \left(A_0 X_A, A_0 X_{\mathbf{x}}, X_{\mathbf{\Pi}}, X_{\boldsymbol{p}}\right),$$

where $(X_A, X_{\mathbf{x}}, X_{\mathbf{\Pi}}, X_{\boldsymbol{p}})$ is the image by (4.22) of the vector $(\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p})$.

Linear stability and instability of the orbitron relative equilibria. The results presented in the previous paragraph provide all the necessary tools to carry out the linear stability analysis of the relative equilibria of the standard and generalized orbitron introduced in the parts **(ii)** and **(iii)** of Proposition 3.1. We will proceed by using expressions (4.21) and (4.22) in order to compute the linearization at the relative equilibria of the Hamiltonian vector fields associated to the augmented Hamiltonians constructed with the appropriate relative equilibrium velocities. We subsequently use part **(v)** of Proposition 4.12 in order to write down the linearization of the restriction of this vector field to the tangent space to a symplectic slice (equivalently, a stability space); finally, we use the last part of this result in order to search for instability regions by looking for eigenvalues of this linearization that exhibit a nontrivial real part and determine how sharp the nonlinear sufficient stability conditions in Theorem 4.5 are; more specifically, we will see that there might exist relative equilibria that are nonlinearly stable even though the conditions in Theorem 4.5 are not satisfied. A detailed description of this implementation is provided in Appendix 5.7. The following result, formulated using the terminology introduced in Proposition 3.1, summarizes the results of the linear analysis.

Theorem 4.14 *Consider the relative equilibria introduced in Proposition 3.1. Then:*

(i) *Regarding the relative equilibria of the standard orbitron in part* **(ii)** *of the proposition:*

- **(a)** *The regular relative equilibria that do not satisfy the Kozorez relation ($r^2/h^2 < 4$) are unstable and this stability condition is consequently sharp. The other two stability conditions in (4.2) and (4.3) are not sharp, that is, there are regions in parameter space that do not satisfy them and where the linearized system is spectrally stable.*
- **(b)** *The singular relative equilibria of the standard orbitron are nonlinearly unstable.*

(ii) *Regarding the relative equilibria of the generalized orbitron in part* **(iii)** *of the proposition:*

- **(a)** *The regular relative equilibria that do not satisfy the generalized Kozorez relation (4.5), namely, $\mu_m \left(2f_1' + r^2 f_2''\right) < 0$, are unstable and this stability condition is consequently sharp. The remaining stability conditions (4.4), (4.6), and (4.7) are not sharp, that is, there are regions in parameter space that do not satisfy them and where the linearized system is spectrally stable.*
- **(b)** *The spectral stability of the singular relative equilibria of the generalized orbitron is equivalent to the following three conditions:*

$$\mu_m f_1' < 0, \tag{4.23}$$

$$\mu_m f_2'' < 0, \tag{4.24}$$

$$\Pi_0^2 > -4\mu_m f_0 I_1, \tag{4.25}$$

where $\Pi_0 = I_3(\xi_1 - \xi_2)$. This statement implies that the nonlinear stability conditions (4.8) and (4.9) are sharp and that the remaining conditions are not.

Proof. **(i) Part (a)** The linearization X_Q at the regular relative equilibria of the Hamiltonian vector field in the stability space associated to the augmented Hamiltonian is provided in the expression (5.92). This matrix is block diagonal and the top two by two block has as eigenvalues

$$\lambda_\pm = \pm \xi_1^0 \sqrt{\frac{r^2 - 4h^2}{r^2 + h^2}},$$

which are real whenever $r^2/h^2 > 4$, that is, when the Kozorez relation is violated. In conclusion, the part **(vi)** of Proposition 4.12 ensures that as soon as the Kozorez relation is violated the relative equilibria cease to be stable. The lack of sharpness of the two other stability conditions in (4.2) and (4.3) is observed by studying the spectrum of the remaining six by six block of the linearization X_Q which may be purely imaginary in regions of the parameter space in which those conditions are violated. The expressions corresponding to those six eigenvalues are very convoluted and we therefore do not include them in the paper; in turn, we illustrate this phenomenon in Figure 3 of the next paragraph, where for a given standard orbitron, we plot the maximum absolute value of the real part of the eigenvalues of the linearization versus the radius of spatial rotation r and the body rotation velocity ξ_2, respectively, when all the system parameters specified in the caption remain constant. The graph on the left hand side shows that when the radius goes beyond the critical value stipulated by the left inequality in (4.2) the spectrum of the linearization remains purely imaginary for a while and the system is hence potentially stable; it is also visible that, as we proved above, the system becomes spectrally unstable as soon as the Kozorez relation ceases to be satisfied. The lack of sharpness of the condition (4.3) is illustrated in the right hand side graph and is of a slight different nature; indeed, as soon as the condition is not satisfied, spectral instability appears but if the body rotation velocity is sufficiently decreased the system becomes again spectrally stable in some interval of the ξ_2 parameter space.

(i) Part (b) The corresponding linearization X_Q at the singular relative equilibria is described in (5.93). Its spectrum includes the two following eigenvalues:

$$\begin{aligned}\lambda_1 &= \frac{1}{h^2}\sqrt{-\frac{3\mu_0\mu_m q}{M\pi}},\\ \lambda_2 &= \sqrt{-\left(\xi_1-\frac{1}{h^2}\sqrt{-\frac{3\mu_0\mu_m q}{2M\pi}}\right)^2}.\end{aligned}$$

The eigenvalue λ_1 can only be purely imaginary when $\mu_m q > 0$. This in turns implies that the term $\sqrt{-\frac{3\mu_0\mu_m q}{2M\pi}}$ in λ_2 is purely imaginary and prevents the eigenvalue to be purely imaginary unless $-\frac{3\mu_0\mu_m q}{2M\pi}$ is zero.

(ii) Part (a) Analogously to the situation in the proof of **(i)** Part **(a)**, the linearization X_Q at the regular relative equilibria of the generalized orbitron exhibits the following two eigenvalues:

$$\lambda_\pm = \pm 2\sqrt{\frac{1}{M}\mu_m\left(2f_1'+r^2f_1''\right)},$$

which are obviously purely imaginary if and only if the generalized Kozorez relation (4.5) holds. The lack of sharpness in the remaining relations follows from the fact that they contain as particular cases the stability conditions for the standard orbitron that, as we illustrate in Figure 3, are not necessary for the spectral stability of X_Q.

(ii) Part (b) The linearization X_Q at the singular relative equilibria of the generalized orbitron is provided in (5.94) and its spectrum is made up by the following ten eigenvalues:

$$\lambda_1^\pm = \pm\sqrt{\frac{1}{M}\mu_m f_2''}, \tag{4.26}$$

$$\lambda_{2,\pm}^\pm = \pm\sqrt{-\frac{1}{M}\left(\xi_1\sqrt{M}\pm\sqrt{-2\mu_m f_1'}\right)^2}, \tag{4.27}$$

$$\lambda_{3,\pm}^\pm = \pm\frac{1}{2}\sqrt{-\frac{1}{I_1}\left((2\xi_1 I_1-\Pi_0)\pm\sqrt{4\mu_m f_0 I_1+\Pi_0^2}\right)^2}. \tag{4.28}$$

The eigenvalues $\lambda_1^\pm$ can be purely imaginary only when $\mu_m f_2'' < 0$. In order for the four eigenvalues $\lambda_{2,\pm}^\pm$ to have the same property, the term $\sqrt{-2\mu_m f_1'}$ has to be necessarily a real number, which yields the condition $\mu_m f_1' < 0$. These two relations obviously imply that the nonlinear stability conditions (4.8) and (4.9) are sharp. Finally, the remaining four eigenvalues $\lambda_{3,\pm}^\pm$ are purely imaginary whenever the term $\sqrt{4\mu_m f_0 I_1+\Pi_0^2}$ is real, which requires in turn that the relation $\Pi_0^2 > 4\mu_m f_0 I_1$ is satisfied. We note that this relation may hold without (4.10) and (4.11) or (4.12) being satisfied. Indeed, take for example a system for which $\mu_m f_0 < 0$; in that situation, the relation (4.25) does not impose any constraint on Π_0 an hence it is easy to find values for this variable that violate (4.10) and (4.11) or (4.12). ∎

Stability transitions and bifurcation analysis. In order to study how spectral and nonlinear stability arises we consider a specific standard orbitron with parameter values $h = 0.05$ m, $M = 0.0068$ kg, $\mu_0 = 4\pi \cdot 10^{-7}$ N·A^{-2}, $\mu = -0.18375$ A·m^2, $q = 17.58$ A·m, $I_1 = 0.17 \cdot 10^{-6}$ kg·m^2, $I_3 = 0.1 \cdot 10^{-6}$ kg·m^2 and analyze the behavior of the eigenvalues of the linearization as a function of the radius r of the spatial rotation and the body rotation velocity ξ_2, in terms of which the nonlinear stability conditions (4.2) and (4.3) are formulated. The results are presented in Figures 3, 4, and 5 whose content we now comment:

- Figure 3 shows the evolution of the maximum value of the real part of the spectrum of the linearization as a function of the radius r of the spatial rotation (left) and the body rotation velocity ξ_2 (right). When this value is zero the whole spectrum obviously lies in the imaginary axis and the system is spectrally stable. The OX-axis in the left hand side figure contains two red dots that indicate the values that determine the radii interval in which the stability condition (4.2) is satisfied; these dots correspond to the values r_{min} and r_{max} in Figure 2. The behavior in r_{max} illustrates the sharpness of the Kozorez stability relation that we established in Theorem 4.14, while for r_{min} it is visible how the system continues for a while to be spectrally stable beyond that value, which creates a gap (grey band II) in which we can neither prove nonlinear stability nor instability with our methods. In the right hand side figure we carry out the same analysis in terms of the body rotation velocity ξ_2 in order to study the sharpness of the condition (4.3); the minimal velocity given by this inequality is indicated using also a red dot in the OX-axis. In this case there is also a gap (grey band II) in which the system is spectrally stable while we do not have definiteness of the stability form.

- Figure 4 depicts the behavior of the spectrum of the linearization as a function of the radius r of the spatial rotation in the different stability regions (I, II, and III) and critical points (A, B, and C) identified in the left hand side of Figure 3 . In region I the system is spectrally unstable until in point A a Hamiltonian Hopf bifurcation (1-1 resonance) [vdM85] takes place and the system becomes spectrally stable in region II. A passing of two eigenvalues takes place in point B and the system becomes nonlinearly stable in region III, as condition (4.2) is satisfied all the way until point C in which a splitting at the origin occurs and the system ceases to be spectrally (and hence nonlinearly) stable. We emphasize that all these eigenvalue patterns are generic in the symmetric Hamiltonian context [DMM92].

- In Figure 5 we carry out the same analysis with respect to the body rotation velocity ξ_2 and we show that similar phenomena occur. More explicitly, in region I the system is spectrally unstable and becomes (only spectrally) stable after a Hamiltonian Hopf bifurcation takes place at point A. This stability is lost in point B through a splitting of eigenvalues at the origin. The figure shows how several splittings take place in region III until in point C the system becomes nonlinearly stable because condition (4.3) is satisfied beyond that point.

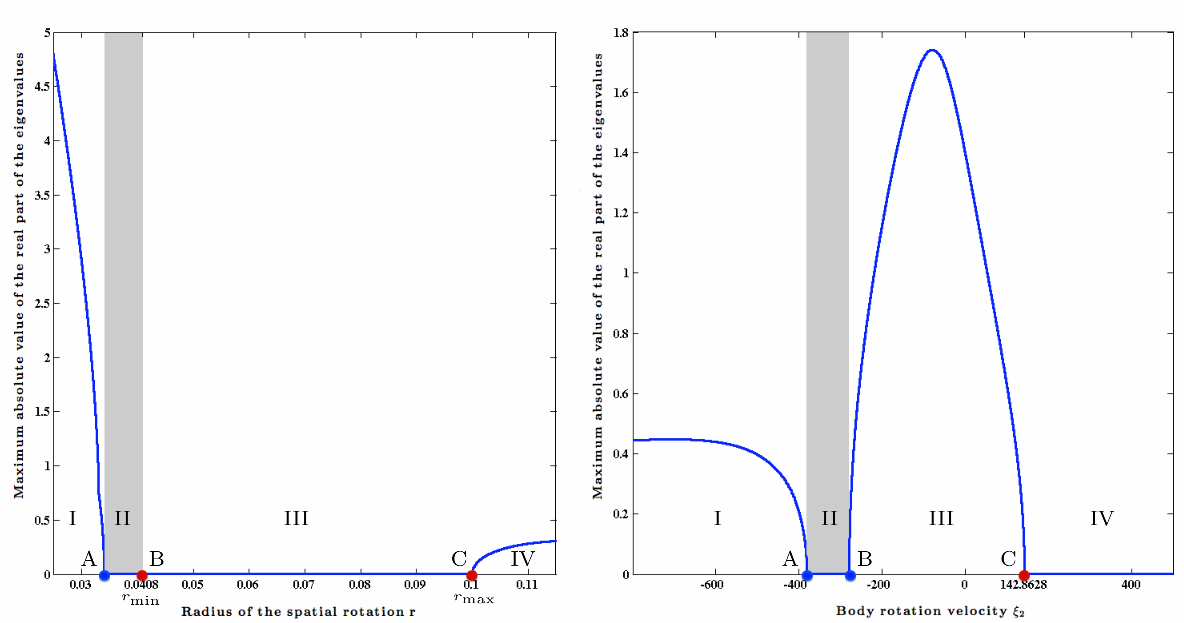

Figure 3: Spectral stability study for the relative equilibria of a standard orbitron with $h = 0.05$ m, $M = 0.0068$ kg, $\mu_0 = 4\pi \cdot 10^{-7}$ N·A^{-2}, $\mu = -0.18375$ A·m^2, $q = 17.58$ A·m, $I_1 = 0.17 \cdot 10^{-6}$ kg·m^2, $I_3 = 0.1 \cdot 10^{-6}$ kg·m^2. The position of the red bullets indicates the critical values of r_{min} and r_{max} (m) and ξ_2 (rad·s^{-1}) determined by the stability conditions (4.2) and (4.3), respectively. The grey bands correspond to the stability gaps discussed in the proof of Theorem 4.14, part **(i)** in which the system is spectrally stable while the stability form exhibits a nontrivial signature.

5 Appendices

5.1 The geometry of the phase space of the orbitron $(T^*SE(3), \omega)$

Lie group and Lie algebra structure of the configuration space. The configuration space of the orbitron is the Lie group $SE(3) = SO(3) \times \mathbb{R}^3$ endowed with the semidirect product structure associated to the composition rule

$$\begin{array}{rccc} \Psi: & SE(3) \times SE(3) & \longrightarrow & SE(3) \\ & ((A_1, \mathbf{x}_1), (A_2, \mathbf{x}_2)) & \longmapsto & (A_1 A_2, A_1 \mathbf{x}_2 + \mathbf{x}_1), \end{array} \tag{5.1}$$

for which $e = (I, 0)$ and $(A, \mathbf{x})^{-1} = (A^{-1}, -A^{-1}\mathbf{x})$. In order to spell out the Lie algebra structure associated to the Lie product (5.1) we start by recalling the Lie algebra isomorphism $\widehat{\ }\colon \left(\mathbb{R}^3, \times\right) \longrightarrow (\mathfrak{so}(3), [\cdot,\cdot])$ between the Lie algebra $(\mathfrak{so}(3), [\cdot,\cdot])$ of $SO(3)$ and $(\mathbb{R}^3, \times)$ endowed with the standard cross product, given by the assignment

$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \longmapsto \widehat{\mathbf{x}} := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

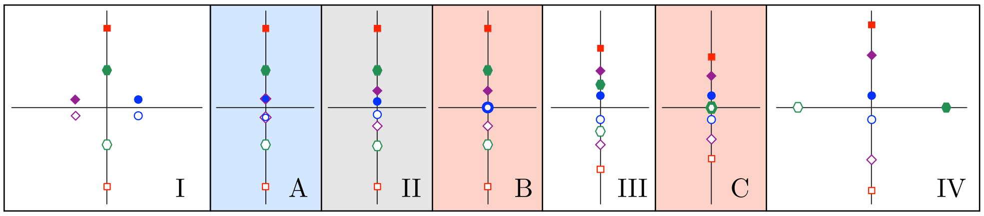

Figure 4: Behavior of the spectrum of the linearization in terms of the radius r of spatial rotation in the different stability regions identified in the left hand side of Figure 3.

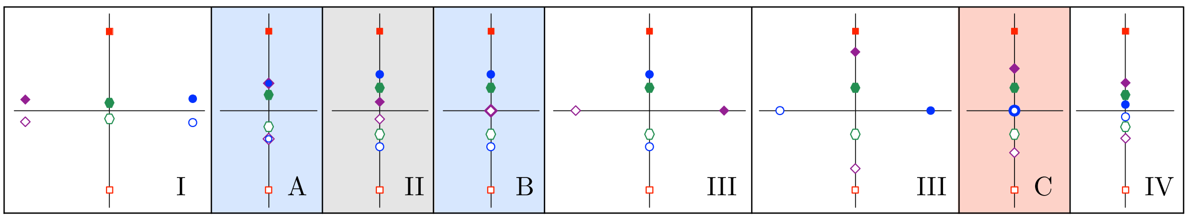

Figure 5: Behavior of the spectrum of the linearization in terms of the body rotation velocity ξ_2 in the different stability regions identified in the right hand side of Figure 3.

We recall that isomorphism $\widehat{\ }$ satisfies that $\widehat{\mathbf{x}}\mathbf{w} = \mathbf{x} \times \mathbf{w}$ and that for any $A \in SO(3)$ and $\mathbf{x} \in \mathbb{R}^3$

$$T_I L_A \cdot \widehat{\mathbf{x}} = A\widehat{\mathbf{x}}, \tag{5.2}$$

$$\mathrm{Ad}_A \widehat{\mathbf{x}} = A\widehat{\mathbf{x}}A^{-1} = \widehat{A\mathbf{x}}, \tag{5.3}$$

$$\mathrm{Ad}_A \widehat{\mathbf{x}} = T_I \left(L_A \circ R_{A^{-1}}\right) \widehat{\mathbf{x}} = \left.\frac{d}{dt}\right|_0 A \exp t\widehat{\mathbf{x}} A^{-1} = A\widehat{\mathbf{x}}A^{-1} = \widehat{A\mathbf{x}}, \tag{5.4}$$

where $L_A : SO(3) \to SO(3)$ (respectively R_A) denotes left (respectively right) translations and $\mathrm{Ad}_A : \mathfrak{so}(3) \to \mathfrak{so}(3)$ is the adjoint representation. The $\widehat{\ }$ isomorphism induces another one

$$\begin{aligned} \widehat{\ } : \mathbb{R}^3 &\longrightarrow \mathfrak{so}(3)^* \\ \boldsymbol{\pi} &\longmapsto \widehat{\boldsymbol{\pi}} \end{aligned}$$

uniquely determined by the relation $\langle \widehat{\boldsymbol{\pi}}, \widehat{\mathbf{x}} \rangle := \langle \boldsymbol{\pi}, \mathbf{x} \rangle_{\mathbb{R}^3}$, with $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ the Euclidean inner product in $\mathbb{R}^3$. Using this isomorphism, we have

$$\mathrm{Ad}^*_A \widehat{\boldsymbol{\pi}} = \widehat{A^{-1}\boldsymbol{\pi}}. \tag{5.5}$$

Using this notation, the Lie algebra structure of $\mathfrak{se}(3) = \mathfrak{so}(3) \times \mathbb{R}^3$ is given by the bracket

$$\left[\left(\widehat{\boldsymbol{\rho}_1}, \boldsymbol{\tau}_1\right), \left(\widehat{\boldsymbol{\rho}_2}, \boldsymbol{\tau}_2\right)\right] := \left(\widehat{\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2}, \boldsymbol{\rho}_1 \times \boldsymbol{\tau}_2 - \boldsymbol{\rho}_2 \times \boldsymbol{\tau}_1\right). \tag{5.6}$$

Additionally, for any $(A,\mathbf{x}) \in SE(3)$, $(\widehat{\boldsymbol{\rho}},\boldsymbol{\tau})\,,(\widehat{\boldsymbol{\rho}}_1,\boldsymbol{\tau}_1)\,,(\widehat{\boldsymbol{\rho}}_2,\boldsymbol{\tau}_2) \in \mathfrak{se}(3)$, $(\widehat{\boldsymbol{\mu}},\boldsymbol{\alpha}) \in \mathfrak{se}(3)^*$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathbb{R}^3$, the following relations that we use later on in the paper hold

$$T_{(I,0)}L_{(A,\mathbf{x})}\cdot(\widehat{\boldsymbol{\rho}},\boldsymbol{\tau}) = (A\widehat{\boldsymbol{\rho}},A\boldsymbol{\tau}) \tag{5.7}$$

$$T_{(I,0)}R_{(A,\mathbf{x})}\cdot(\widehat{\boldsymbol{\rho}},\boldsymbol{\tau}) = (\widehat{\boldsymbol{\rho}}A,\boldsymbol{\rho}\times\mathbf{x}+\boldsymbol{\tau}) \tag{5.8}$$

$$\mathrm{ad}_{(\widehat{\boldsymbol{\rho}_1},\boldsymbol{\tau}_1)}(\widehat{\boldsymbol{\rho}}_2,\boldsymbol{\tau}_2) = \left(\widehat{\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2},\boldsymbol{\rho}_1\times\boldsymbol{\tau}_2-\boldsymbol{\rho}_2\times\boldsymbol{\tau}_1\right), \tag{5.9}$$

$$\mathrm{ad}^*_{(\widehat{\boldsymbol{\rho}},\boldsymbol{\tau})}(\widehat{\boldsymbol{\mu}},\boldsymbol{\alpha}) = \left(\widehat{\boldsymbol{\mu}\times\boldsymbol{\rho}}+\widehat{\boldsymbol{\alpha}\times\boldsymbol{\tau}},\boldsymbol{\alpha}\times\boldsymbol{\rho}\right), \tag{5.10}$$

$$T^*_{(I,0)}R_{(A,\mathbf{x})}\left(\widehat{\boldsymbol{\beta}}A,\boldsymbol{\gamma}\right) = (\boldsymbol{\beta}+\mathbf{x}\times\boldsymbol{\gamma},\boldsymbol{\gamma})\,, \tag{5.11}$$

$$T^*_{(I,0)}L_{(A,\mathbf{x})}\left(A\widehat{\boldsymbol{\beta}},\boldsymbol{\gamma}\right) = (\boldsymbol{\beta},A^{-1}\boldsymbol{\gamma})\,. \tag{5.12}$$

In the last two expressions we have identified $T^*_{(A,\mathbf{x})}SE(3)$ with $T_{(A,\mathbf{x})}SE(3)$ using the Frobenius norm in the $SO(3)$ part and the Euclidean norm in the $\mathbb{R}^3$ part. Using these equalities, it is easy to see that the adjoint and coadjoint actions of $SE(3)$ on its algebra $\mathfrak{se}(3)$ and its dual $\mathfrak{se}(3)^*$ are determined by:

$$\mathrm{Ad}_{(A,\mathbf{x})}(\widehat{\boldsymbol{\rho}},\boldsymbol{\tau}) = (\mathrm{Ad}_A\widehat{\boldsymbol{\rho}},-(\mathrm{Ad}_A\widehat{\boldsymbol{\rho}})\mathbf{x}+A\boldsymbol{\tau}) = \left(\widehat{A\boldsymbol{\rho}},\mathbf{x}\times A\boldsymbol{\rho}+A\boldsymbol{\tau}\right), \tag{5.13}$$

$$\mathrm{Ad}^*_{(A,\mathbf{x})}(\widehat{\boldsymbol{\mu}},\boldsymbol{\alpha}) = \left(\mathrm{Ad}^*_A\widehat{\boldsymbol{\mu}}-(A^{-1}\widehat{(\mathbf{x}\times\boldsymbol{\alpha})}),A^{-1}\boldsymbol{\alpha}\right) = \left(\left(A^{-1}(\widehat{\boldsymbol{\mu}-(\mathbf{x}}\times\boldsymbol{\alpha}))\right),A^{-1}\boldsymbol{\alpha}\right). \tag{5.14}$$

Body and space coordinates for $T^*SE(3)$. Given an arbitrary Lie group G with Lie algebra $\mathfrak{g}$, we recall (see for example [AM78]) that the maps

$$\begin{array}{cccc} \varrho_1: & TG & \longrightarrow & G\times g \\ & u_g & \longmapsto & (g,T_gR_{g^{-1}}\cdot u_g) \\ & T_eR_g\cdot\xi & \longmapsto & (g,\xi). \end{array} \quad \text{and} \quad \begin{array}{cccc} \varrho_2: & T^*G & \longrightarrow & G\times g^* \\ & \alpha_g & \longmapsto & (g,T_e^*R_g\cdot\alpha_g) \\ & T_g^*R_{g^{-1}}\cdot\mu & \longmapsto & (g,\mu) \end{array} \tag{5.15}$$

define trivializations of the tangent TG and cotangent bundles T^*G, respectively, that are usually referred to as ***space coordinates*** of these bundles. Notice that if $\varrho_1(u_g) = (g,\xi)$, $\varrho_2(\alpha_g) = (g,\mu)$, then $\langle\alpha_g,u_g\rangle = \langle\mu,\xi\rangle$.

Analogously, the trivializations obtained using left translations instead via the maps

$$\begin{array}{cccc} \lambda_1: & TG & \longrightarrow & G\times g \\ & u_g & \longmapsto & (g,T_gL_{g^{-1}}\cdot u_g) \\ & T_eL_g\cdot\xi & \longmapsto & (g,\xi). \end{array} \quad \text{and} \quad \begin{array}{cccc} \lambda_2: & T^*G & \longrightarrow & G\times g^* \\ & \alpha_g & \longmapsto & (g,T_e^*L_g\cdot\alpha_g) \\ & T_g^*L_{g^{-1}}\cdot\mu & \longmapsto & (g,\mu) \end{array} \tag{5.16}$$

are usually referred to as ***body coordinates***. Notice that if $\lambda_1(u_g) = (g,\xi)$, $\lambda_2(\alpha_g) = (g,\mu)$, then $\langle\alpha_g,u_g\rangle = \langle\mu,\xi\rangle$.

We now use these maps to establish the relation between the space and body coordinates $((A,\mathbf{x}),(\boldsymbol{\Pi}_S,\mathbf{p}_S))$ and $((A,\mathbf{x}),(\boldsymbol{\Pi}_B,\mathbf{p}_B))$, respectively, of an arbitrary point in $T^*SE(3)$. Indeed, using (5.16), (5.15), and (5.14), we have that

$$\begin{aligned} ((A,\mathbf{x}),(\boldsymbol{\Pi}_B,\mathbf{p}_B)) &= \lambda_2\left(T^*_{(A,\mathbf{x})}R_{(A,\mathbf{x})^{-1}}\cdot(\boldsymbol{\Pi}_S,\mathbf{p}_S)\right) = \left((A,\mathbf{x}),\mathrm{Ad}^*_{(A,\mathbf{x})}(\boldsymbol{\Pi}_S,\mathbf{p}_S)\right) \\ &= \left((A,\mathbf{x}),(A^{-1}(\boldsymbol{\Pi}_S-\mathbf{x}\times\mathbf{p}_S),A^{-1}\mathbf{p}_S)\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \boldsymbol{\Pi}_B &= A^{-1}(\boldsymbol{\Pi}_S-\mathbf{x}\times\mathbf{p}_S)\,, \\ \mathbf{p}_B &= A^{-1}\mathbf{p}_S. \end{aligned} \tag{5.17}$$

Conversely,

$$
\begin{aligned}
\mathbf{\Pi}_S &= A\mathbf{\Pi}_B + \mathbf{x} \times A\mathbf{p}_B, \\
\mathbf{p}_S &= A\mathbf{p}_B.
\end{aligned} \tag{5.18}
$$

5.2 Equations of motion of the orbitron

In this section we obtain the equations of motion (2.11)-(2.14) of the orbitron using body coordinates. We will proceed by writing down first the differential equations that define a Hamiltonian vector field on the left trivialized cotangent bundle $G \times \mathfrak{g}^*$ of an arbitrary Lie group G with Lie algebra $\mathfrak{g}$.

Proposition 5.1 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let T^*G be its cotangent bundle endowed with the canonical symplectic form. Let ω_B be the corresponding symplectic form on the trivial bundle $G \times \mathfrak{g}^*$ obtained out of T^*G by left trivialization (body coordinates) and let $H \in C^\infty(G \times \mathfrak{g}^*)$ be a Hamiltonian function. For any $(g, \mu) \in G \times \mathfrak{g}^*$, the Hamiltonian vector field $X_H \in \mathfrak{X}(G \times \mathfrak{g}^*)$ associated to H is given by*

$$
X_H(g,\mu) = \left(T_I L_g \cdot X_G(g,\mu), X_{\mathfrak{g}^*}(g,\mu)\right), \tag{5.19}
$$

where $X_G(g,\mu) \in \mathfrak{g}$ and $X_{\mathfrak{g}^*}(g,\mu) \in \mathfrak{g}^*$ are determined by

$$
X_G(g,\mu) = D_{\mathfrak{g}^*}H(g,\mu), \tag{5.20}
$$

$$
X_{\mathfrak{g}^*}(g,\mu) = -T_I^* L_g \cdot D_G H(g,\mu) + \mathrm{ad}^*_{D_{\mathfrak{g}^*}H(g,\mu)}\mu. \tag{5.21}
$$

Proof. Using the expression of the canonical symplectic form ω_B of T^*G in body coordinates (see for instance [OR04, Expression (6.2.5)]) it is easy to see that X_G, $X_{\mathfrak{g}^*}$, and hence X_H, are determined by the relation

$$
\begin{aligned}
\omega_B(g,\mu)\left(X_H(g,\mu), (T_I L_g \cdot \xi_G, \beta)\right) &= \langle \beta, X_G(g,\mu)\rangle - \langle X_{g^*}(g,\mu), \xi_G\rangle \\
&\quad + \langle \mu, [X_G(g,\mu), \xi_G]\rangle = D_G H(g,\mu) \cdot T_I L_g \cdot \xi_G + D_{\mathfrak{g}^*}H(g,\mu)\cdot\beta,
\end{aligned}
$$

where $\xi_G \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$ are arbitrary and $D_G H$ and $D_{\mathfrak{g}^*}H$ are the partial derivatives of H with respect to G and $\mathfrak{g}^*$, respectively. Equivalently,

$$
\begin{aligned}
X_G(g,\mu) &= D_{g^*}H(g,\mu), \\
X_{g^*}(g,\mu) &= -T_I^* L_g \cdot D_G H(g,\mu) + \mathrm{ad}^*_{D_{g^*}H(g,\mu)}\mu,
\end{aligned}
$$

as required. ■

We now consider the case we are interested in, that is, $G = SE(3) = SO(3) \times \mathbb{R}^3$ and

$$
H\left((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})\right) = \frac{1}{2}\mathbf{\Pi}^T \mathbb{I}_{\mathrm{ref}}^{-1}\mathbf{\Pi} + \frac{1}{2M}\mathbf{p}^T\mathbf{p} - \mu_m\langle \mathbf{B}(\mathbf{x}), A\mathbf{e}_3\rangle. \tag{5.22}
$$

Let

$$
v_{(A,\mathbf{x})} = T_{(I,\mathbf{0})}L_{(A,\mathbf{x})} \cdot \left(\widehat{\delta A}, \boldsymbol{\delta}\mathbf{x}\right) = \left(A\widehat{\delta A}, A\boldsymbol{\delta}\mathbf{x}\right)
$$

be an arbitrary element of $T_{(A,\mathbf{x})}SE(3)$ and $\beta = (\boldsymbol{\delta}\mathbf{\Pi}, \boldsymbol{\delta}\mathbf{p}) \in \mathfrak{se}(3)^*$. Then, as

$$
\begin{aligned}
\mathbf{d}H\left((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})\right)\cdot\left(v_{(A,\mathbf{x})},\beta\right) &= \left.\frac{d}{dt}\right|_0 H\left(\left((A,\mathbf{x})\cdot(\exp t\widehat{\delta A}, t\boldsymbol{\delta}\mathbf{x})\right), (\mathbf{\Pi}+t\boldsymbol{\delta}\mathbf{\Pi}, \mathbf{p}+t\boldsymbol{\delta}\mathbf{p})\right) = \\
&\langle \mathbb{I}_{\mathrm{ref}}^{-1}\mathbf{\Pi}, \boldsymbol{\delta}\mathbf{\Pi}\rangle + \frac{1}{M}\langle \mathbf{p}, \boldsymbol{\delta}\mathbf{p}\rangle - \mu_m\langle DB(\mathbf{x})^T A\mathbf{e}_3, A\boldsymbol{\delta}\mathbf{x}\rangle + \langle A(\mathbf{e}_3 \times \widehat{A^{-1}\mathbf{B}(\mathbf{x})}), A\widehat{\delta A}\rangle,
\end{aligned}
$$

we can conclude that

$$D_G H((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})) = \begin{pmatrix} A\left[\mathbf{e}_3 \times \widehat{A^{-1}\mathbf{B}(\mathbf{x})}\right] \\ -\mu_m D\mathbf{B}(\mathbf{x})^T A\mathbf{e}_3 \end{pmatrix}, \tag{5.23}$$

$$D_{g^*} H((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})) = \begin{pmatrix} \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} \\ \frac{1}{M}\mathbf{p} \end{pmatrix}. \tag{5.24}$$

Now using (5.20) and (5.21), together with and (5.23), (5.24), (5.10), and (5.12), we obtain

$$X_{g^*}(g,\mu) = \begin{pmatrix} -\mathbf{e}_3 \times A^{-1}\mathbf{B}(\mathbf{x}) + \mathbf{\Pi} \times \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} \\ \mu_m A^{-1} D\mathbf{B}(\mathbf{x})^T A\mathbf{e}_3 + \mathbf{p} \times \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} \end{pmatrix},$$

$$X_G(g,\mu) = \begin{pmatrix} \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} \\ \frac{1}{M}\mathbf{p} \end{pmatrix}.$$

Consequently, by (5.19) we conclude that the equations of motion associated to the Hamiltonian (5.22) are

$$\begin{aligned} \dot{A} &= A\widehat{\mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi}}, \\ \dot{\mathbf{x}} &= \frac{1}{M}A\mathbf{p}, \\ \dot{\mathbf{\Pi}} &= \mathbf{\Pi} \times \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} + A^{-1}\mathbf{B}(\mathbf{x}) \times \mathbf{e}_3, \\ \dot{\mathbf{p}} &= \mathbf{p} \times \mathbb{I}_{\text{ref}}^{-1}\mathbf{\Pi} + \mu_m A^{-1} D\mathbf{B}(\mathbf{x})^T A\mathbf{e}_3. \end{aligned}$$

5.3 The toral action on phase space $T^*SE(3)$ and the associated momentum map

The expression of the lifted action in body coordinates. We start by proving that the cotangent lift of the toral action on $SE(3)$ in (2.15) is given by (2.16) when using body coordinates. Consider H and G two arbitrary Lie groups and let $\Phi : H \times G \longrightarrow G$ be an action of H on G. We recall that the lift of this action to the cotangent bundle T^*G of G, also denoted by Φ, is given by

$$\begin{array}{cccc} \Phi: & H \times T^*G & \longrightarrow & T^*G \\ & (h, \alpha_g) & \longmapsto & T^*_{\Phi_h(g)}\Phi_{h^{-1}} \cdot \alpha_g. \end{array}$$

Using the maps introduced in (5.16), this action is expressed in body coordinates as:

$$\Phi(h,(g,\mu)) := \lambda_2\left(\Phi\left(h, \lambda_2^{-1}(g,\mu)\right)\right), \quad \text{for any} \quad h \in H, g \in G, \text{ and } \mu \in \mathfrak{g}^*,$$

or equivalently,

$$\Phi_h(g,\mu) = \left(\Phi_h(g), T_e^* L_{\Phi_h(g)} \cdot T^*_{\Phi_h(g)}\left(L_{g^{-1}} \circ \Phi_{h^{-1}}\right)\mu\right) = \left(\Phi_h(g), T_e^*\left(L_{g^{-1}} \circ \Phi_{h^{-1}} \circ L_{\Phi_h(g)}\right)\mu\right).$$

In the particular case of $H = \mathbb{T}^2$, $G = SE(3)$, and the toral action introduced in (2.15), that is,

$$\begin{array}{cccc} \Phi: & (\mathbb{T}^2 = S^1 \times S^1) \times SE(3) & \longrightarrow & SE(3) \\ & \left(\left(e^{i\theta_S}, e^{i\theta_B}\right), (A,\mathbf{x})\right) & \longmapsto & (R^Z_{\theta_S} A R^Z_{-\theta_B}, R^Z_{\theta_S}\mathbf{x}), \end{array}$$

we consider $g = (A,\mathbf{x}) \in SE(3)$, $\mu = \left(\widehat{\mathbf{\Pi}}, \mathbf{p}\right) \in \mathfrak{se}(3)^*$, and $h = \left(e^{i\theta_S}, e^{i\theta_B}\right) \in \mathbb{T}^2$. Then,

$$\Phi_h(g,\mu) = \left(\left(R^Z_{\theta_S} A R^Z_{\theta_{-B}}, R^Z_{\theta_S}\mathbf{x}\right), T_e^*\left(L_{g^{-1}} \circ \Phi_{h^{-1}} \circ L_{\Phi_h(g)}\right)\mu\right). \tag{5.25}$$

In order to compute the second part of this expression let $\xi = (\widehat{\boldsymbol{\rho}}, \boldsymbol{\tau}) \in \mathfrak{se}(3)$. Then

$$
\begin{aligned}
&\left\langle T_e^* \left(L_{g^{-1}} \circ \Phi_{h^{-1}} \circ L_{\Phi_h(g)}\right) \mu, \xi \right\rangle = \left\langle \mu, T_e \left(L_{g^{-1}} \circ \Phi_{h^{-1}} \circ L_{\Phi_h(g)}\right) \xi \right\rangle \\
&= \frac{d}{dt}\bigg|_0 \left\langle \left(\widehat{\boldsymbol{\Pi}}, \mathbf{p}\right), L_{(A^{-1}, -A^{-1}\mathbf{x})} \circ \Phi_{\left(e^{-i\theta_S}, e^{-i\theta_B}\right)} \circ L_{\left(R^Z_{\theta_S} A R^Z_{-\theta_B}, R^Z_{\theta_S}\mathbf{x}\right)} (\exp t\widehat{\boldsymbol{\rho}}, t\boldsymbol{\tau}) \right\rangle \\
&= \frac{d}{dt}\bigg|_0 \left\langle \left(\widehat{\boldsymbol{\Pi}}, \mathbf{p}\right), \left(A^{-1} A R^Z_{-\theta_B} \exp t\widehat{\boldsymbol{\rho}} R^Z_{\theta_B}, A^{-1} A R^Z_{-\theta_B} t\boldsymbol{\tau} + A^{-1}\mathbf{x} - A^{-1}\mathbf{x}\right) \right\rangle \\
&= \frac{d}{dt}\bigg|_0 \left\langle \left(\widehat{\boldsymbol{\Pi}}, \mathbf{p}\right), \left(R^Z_{-\theta_B} \exp t\widehat{\boldsymbol{\rho}} R^Z_{\theta_B}, t R^Z_{-\theta_B}\boldsymbol{\tau}\right) \right\rangle = \left\langle \left(\widehat{\boldsymbol{\Pi}}, \mathbf{p}\right), \left(\mathrm{Ad}_{\mathrm{R}^{\mathrm{Z}}_{-\theta_B}} \widehat{\boldsymbol{\rho}}, \mathrm{R}^{\mathrm{Z}}_{-\theta_B}\boldsymbol{\tau}\right) \right\rangle \\
&= \left\langle \left(\mathrm{Ad}^*_{R^Z_{-\theta_B}} \widehat{\boldsymbol{\Pi}}, R^Z_{\theta_B}\mathbf{p}\right), (\widehat{\boldsymbol{\rho}}, \boldsymbol{\tau}) \right\rangle.
\end{aligned}
$$

Given that by (5.4) $\mathrm{Ad}^*{}_{R^Z_{-\theta_B}} \widehat{\boldsymbol{\Pi}} = \widehat{R^Z_{\theta_B}\boldsymbol{\Pi}}$, the last equality together with (5.25) yield the expression (2.16) of the lifted action in body coordinates, that is,

$$
\Phi_{\left(e^{i\theta_S}, e^{i\theta_B}\right)} ((A, \mathbf{x}), (\boldsymbol{\Pi}, \mathbf{p})) = \left((R_{\theta_S} A R^Z_{-\theta_B}, R^Z_{\theta_S}\mathbf{x}), (R^Z_{\theta_B}\boldsymbol{\Pi}, R^Z_{\theta_B}\mathbf{p})\right). \tag{5.26}
$$

The infinitesimal generators of the toral action. We first show that for any Lie algebra element $(\xi_1, \xi_2) \in \mathbb{R}^2 = \mathrm{Lie}(\mathbb{T}^2)$ and $(A, \mathbf{x}) \in SE(3)$,

$$
\begin{aligned}
(\xi_1, \xi_2)_{SE(3)} (A, \mathbf{x}) &= T_{(I,0)} R_{(A,\mathbf{x})} \left(\widehat{\xi_1 \mathbf{e}_3 - A\xi_2\mathbf{e}_3}, A\xi_2\mathbf{e}_3 \times \mathbf{x}\right) &\quad (5.27)\\
&= T_{(I,0)} L_{(A,\mathbf{x})} \left(\mathrm{Ad}_{A^{-1}} \widehat{\xi_1\mathbf{e}_3} - \widehat{\xi_2\mathbf{e}_3}, A^{-1}(\xi_1\mathbf{e}_3 \times \mathbf{x})\right). &\quad (5.28)
\end{aligned}
$$

We start by proving the first equality

$$
\begin{aligned}
&(\xi_1, \xi_2)_{SE(3)} (A, \mathbf{x}) = \frac{d}{dt}\bigg|_0 (\exp t\widehat{\xi_1\mathbf{e}_3} A \exp(-t\widehat{\xi_2\mathbf{e}_3}), \exp t\widehat{\xi_1\mathbf{e}_3}\mathbf{x}) = (\widehat{\xi_1\mathbf{e}_3} A - A\widehat{\xi_2\mathbf{e}_3}, \widehat{\xi_1\mathbf{e}_3}\mathbf{x}) \\
&= (\widehat{\xi_1\mathbf{e}_3} A - A\widehat{\xi_2\mathbf{e}_3} A^{-1} A, \xi_1\mathbf{e}_3 \times \mathbf{x}) = \left((\widehat{\xi_1\mathbf{e}_3} - A\widehat{\xi_2\mathbf{e}_3} A^{-1})A, (\widehat{\xi_1\mathbf{e}_3} - A\widehat{\xi_2\mathbf{e}_3} + A\widehat{\xi_2\mathbf{e}_3}) \times \mathbf{x}\right) \\
&= \left((\widehat{\xi_1\mathbf{e}_3 - A\xi_2\mathbf{e}_3})A, (\xi_1\mathbf{e}_3 - A\xi_2\mathbf{e}_3) \times \mathbf{x} + (A\xi_2\mathbf{e}_3 \times \mathbf{x})\right) = T_{(I,0)} R_{(A,\mathbf{x})} \left((\widehat{\xi_1\mathbf{e}_3 - A\xi_2\mathbf{e}_3}), A\xi_2\mathbf{e}_3 \times \mathbf{x}\right),
\end{aligned}
$$

where in the last equality we used (5.8). Regarding (5.28), note that

$$
\begin{aligned}
&(\xi_1, \xi_2)_{SE(3)} (A, \mathbf{x}) = \frac{d}{dt}\bigg|_0 (\exp t\widehat{\xi_1\mathbf{e}_3} A \exp(-t\widehat{\xi_2\mathbf{e}_3}), \exp t\widehat{\xi_1\mathbf{e}_3}\mathbf{x}) = (\widehat{\xi_1\mathbf{e}_3} A - A\widehat{\xi_2\mathbf{e}_3}, \widehat{\xi_1\mathbf{e}_3}\mathbf{x}) \\
&= (AA^{-1}\widehat{\xi_1\mathbf{e}_3} A - A\widehat{\xi_2\mathbf{e}_3}, \xi_1\mathbf{e}_3 \times \mathbf{x}) = \left(T_I L_A(\mathrm{Ad}_{A^{-1}}\widehat{\xi_1\mathbf{e}_3} - \widehat{\xi_2\mathbf{e}_3}), (AA^{-1}(\xi_1\mathbf{e}_3 \times \mathbf{x}))\right) \\
&= T_{(I,0)} L_{(A,\mathbf{x})} \left(\mathrm{Ad}_{A^{-1}}\widehat{\xi_1\mathbf{e}_3} - \widehat{\xi_2\mathbf{e}_3}, A^{-1}(\xi_1\mathbf{e}_3 \times \mathbf{x})\right),
\end{aligned}
$$

where we used (5.7).

The infinitesimal generator of the lifted $\mathbb{T}^2$-action on $T^*SE(3)$ in body coordinates is given by

$$
(\xi_1, \xi_2)_{T^*SE(3)} (A, \mathbf{x}, \boldsymbol{\Pi}, \mathbf{p}) = \left(A(\mathrm{Ad}_{A^{-1}}(\widehat{\xi_1\mathbf{e}_3}) - \widehat{\xi_2\mathbf{e}_3}), \widehat{\xi_1\mathbf{e}_3}\mathbf{x}, \widehat{\xi_2\mathbf{e}_3}\boldsymbol{\Pi}, \widehat{\xi_2\mathbf{e}_3}\mathbf{p}\right) \tag{5.29}
$$

Indeed,

$$(\xi_1,\xi_2)_{T^*SE(3)}(A,\mathbf{x},\mathbf{\Pi},\mathbf{p}) = \left.\frac{d}{dt}\right|_0 \exp t(\xi_1,\xi_2)\cdot(A,\mathbf{x},\mathbf{\Pi},\mathbf{p})$$
$$= \left.\frac{d}{dt}\right|_0 \left(\exp t\widehat{\xi_1\mathbf{e}_3}A\exp(-t\widehat{\xi_2\mathbf{e}_3}),\exp t\widehat{\xi_1\mathbf{e}_3}\mathbf{x},\exp t\widehat{\xi_2\mathbf{e}_3}\mathbf{\Pi},\exp t\widehat{\xi_2\mathbf{e}_3}\mathbf{p}\right)$$
$$= \left(AA^{-1}\widehat{\xi_1\mathbf{e}_3}A - A\widehat{\xi_2\mathbf{e}_3},\widehat{\xi_1\mathbf{e}_3}\mathbf{x},\widehat{\xi_2\mathbf{e}_3}\mathbf{\Pi},\widehat{\xi_2\mathbf{e}_3}\mathbf{p}\right) = \left(A\left(\mathrm{Ad}_{A^{-1}}\left(\widehat{\xi_1\mathbf{e}_3}\right)-\widehat{\xi_2\mathbf{e}_3}\right),\widehat{\xi_1\mathbf{e}_3}\mathbf{x},\widehat{\xi_2\mathbf{e}_3}\mathbf{\Pi},\widehat{\xi_2\mathbf{e}_3}\mathbf{p}\right).$$

The momentum map of the toral action Given a lifted action of a Lie group H on the cotangent bundle T^*G of a Lie group G endowed with the canonical symplectic form, the map $\mathbf{J}:T^*G\longrightarrow\mathfrak{g}^*$ defined by

$$\langle \boldsymbol{J}(\alpha_g),\xi\rangle = \langle\alpha_g,\xi_G(g)\rangle \quad \text{for any} \quad g\in G,\ \alpha_g\in T^*G,\ \text{and}\ \xi\in\mathfrak{h}, \tag{5.30}$$

is a coadjoint equivariant momentum map for this canonical action (see [AM78, Corollary 4.2.11]). We now study the particular case we are interested in, that is, $H=\mathbb{T}^2$, $G=SE(3)$, and consider an arbitrary point $g=(A,\mathbf{x})\in SE(3)$, $\mu=(\mathbf{\Pi},\mathbf{p})\in\mathfrak{se}(3)^*$ and $\alpha_g=T_g^*L_{g^{-1}}\cdot\mu\in T^*SE(3)$ the covector that in body coordinates is expressed as $((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p}))$. With this notation, the expression in body coordinates of the momentum map $\mathbf{J}:SE(3)\times\mathfrak{se}(3)^*\longrightarrow\mathbb{R}^2$ in (5.30) is given by

$$\boldsymbol{J}((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})) = (\langle A\mathbf{\Pi}+\mathbf{x}\times A\mathbf{p},\mathbf{e}_3\rangle,-\langle\mathbf{\Pi},\mathbf{e}_3\rangle). \tag{5.31}$$

Indeed, for any $(\xi_1,\xi_2)\in\mathbb{R}^2$,

$$\langle\boldsymbol{J}((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})),(\xi_1,\xi_2)\rangle = \left\langle T^*_{(A,\mathbf{x})}L_{(A,\mathbf{x})^{-1}}(\mathbf{\Pi},\mathbf{p}),T_{(I,0)}L_{(A,\mathbf{x})}\left((A^{-1}\widehat{\xi_1\mathbf{e}_3-\xi_2\mathbf{e}_3}),A^{-1}(\xi_1\mathbf{e}_3\times\mathbf{x})\right)\right\rangle$$
$$= \langle\mathbf{\Pi},A^{-1}\xi_1\mathbf{e}_3-\xi_2\mathbf{e}_3\rangle+\langle\mathbf{p},A^{-1}(\xi_1\mathbf{e}_3\times\mathbf{x})\rangle = \langle\mathbf{\Pi},A^{-1}\xi_1\mathbf{e}_3-\xi_2\mathbf{e}_3\rangle+\langle\mathbf{p},A^{-1}(\xi_1\mathbf{e}_3\times\mathbf{x})\rangle,$$

which proves (5.31) since $(\xi_1,\xi_2)\in\mathbb{R}^2$ is arbitrary.

5.4 Proof of Proposition 3.1

(i) Using the statement preceeding (3.3) we will specify the relative equilibria of the orbitron by characterizing the points $\mathbf{z}=((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p}))\in T^*SE(3)$ for which

$$\mathbf{d}(H-\boldsymbol{J}^{(\xi_1,\xi_2)})((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p}))=0 \tag{5.32}$$

for some $(\xi_1,\xi_2)\in\mathbb{R}^2$. We start by computing the tangent of the momentum map and the differential of the Hamiltonian. Let $\mathbf{v}=\left((\widehat{\delta A}A,\boldsymbol{\delta}\mathbf{x}),(\boldsymbol{\delta\Pi},\boldsymbol{\delta}\mathbf{p})\right)\in T_{\mathbf{z}}(T^*SE(3))$ be an arbitrary vector at the point $\mathbf{z}$, then it is easy to check that

$$\mathbf{d}T(\mathbf{\Pi},\mathbf{p})\cdot\mathbf{v} = \langle\mathbf{\Pi},\mathbb{I}_{\mathrm{ref}}^{-1}\boldsymbol{\delta\Pi}\rangle+\frac{1}{M}\langle\mathbf{p},\boldsymbol{\delta}\mathbf{p}\rangle, \tag{5.33}$$
$$\mathbf{d}V(A,\mathbf{x})\cdot\mathbf{v} = -\mu_m[\langle D\mathbf{B}(\mathbf{x})(\boldsymbol{\delta}\mathbf{x}),A\mathbf{e}_3\rangle+\langle\mathbf{B}(\mathbf{x}),\delta A\times A\mathbf{e}_3\rangle]. \tag{5.34}$$

with T and V the kinetic and potential energies introduced in (2.2). Additionally,

$$T_{((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p}))}\boldsymbol{J}\cdot\left((\widehat{\delta A}A,\boldsymbol{\delta}\mathbf{x}),(\boldsymbol{\delta\Pi},\boldsymbol{\delta}\mathbf{p})\right)$$
$$= \left(\langle\delta A\times A\mathbf{\Pi}+A\boldsymbol{\delta\Pi}+\boldsymbol{\delta}\mathbf{x}\times A\mathbf{p}+\mathbf{x}\times(\delta A\times A\mathbf{p})+\mathbf{x}\times A\boldsymbol{\delta}\mathbf{p},\mathbf{e}_3\rangle,-\langle\boldsymbol{\delta\Pi},\mathbf{e}_3\rangle\right). \tag{5.35}$$

Consequently, using (5.33), (5.34) and (5.35) we have, for any $(\xi_1, \xi_2) \in \mathbb{R}^2$

$$\mathbf{d}\left(H - \boldsymbol{J}^{(\xi_1,\xi_2)}\right)(\mathbf{z})\cdot\mathbf{v} = \boldsymbol{\Pi}^T \mathbb{I}_{\rm ref}^{-1}\boldsymbol{\delta\Pi} + \frac{1}{M}\mathbf{p}\cdot\boldsymbol{\delta}\mathbf{p} - \mu_m\left[\langle D\mathbf{B}(\mathbf{x})(\boldsymbol{\delta}\mathbf{x}), A\mathbf{e}_3\rangle + \langle \mathbf{B}(\mathbf{x}), \delta A\times A\mathbf{e}_3\rangle\right]$$
$$+ \xi_2\boldsymbol{\delta\Pi}\cdot\mathbf{e}_3 - \xi_1\left(\delta A\times A\boldsymbol{\Pi} + A\boldsymbol{\delta\Pi} + \boldsymbol{\delta}\mathbf{x}\times A\mathbf{p} + \mathbf{x}\times(\delta A\times A\mathbf{p}) + \mathbf{x}\times A\boldsymbol{\delta}\mathbf{p}\right)\cdot\mathbf{e}_3. \quad (5.36)$$

Therefore, as $\widehat{\delta A}$, $\boldsymbol{\delta}\mathbf{x}$, $\boldsymbol{\delta\Pi}$, and $\boldsymbol{\delta}\mathbf{p}$ in this expression are arbitrary, it can be checked that the points $\mathbf{z}\in T^*SE(3)$ for which $\mathbf{d}(H - \boldsymbol{J}^{(\xi_1,\xi_2)})(\mathbf{z}) = 0$ are characterized by the equations:

$$\mu_m\left[\mathbf{B}(\mathbf{x})\times A\mathbf{e}_3\right] + \xi_1\left[A\mathbf{p}\times(\mathbf{x}\times\mathbf{e}_3) - A\boldsymbol{\Pi}\times\mathbf{e}_3\right] = 0, \quad (5.37)$$
$$-\mu_m D\mathbf{B}(\mathbf{x})^T(A\mathbf{e}_3) - \xi_1\left(A\mathbf{p}\times\mathbf{e}_3\right) = 0, \quad (5.38)$$
$$\mathbb{I}_{\rm ref}^{-1}\boldsymbol{\Pi} + \xi_2\mathbf{e}_3 - \xi_1 A^{-1}\mathbf{e}_3 = 0, \quad (5.39)$$
$$\frac{1}{M}\mathbf{p} - \xi_1 A^{-1}\left(\mathbf{e}_3\times\mathbf{x}\right) = 0, \quad (5.40)$$

as required.

(ii) We show that the points $\mathbf{z}_0 = ((A_0, \mathbf{x}_0), (\boldsymbol{\Pi}_0, \mathbf{p}_0))$ of the form specified in the statement of the proposition satisfy equations (5.37)–(5.40) and hence constitute a branch of relative equilibria. We proceed by considering $A_0 = R^Z_{\theta_0}$ and $\mathbf{x}_0 = (x, y, 0)$ and using equations (5.37)–(5.40) to determine $\boldsymbol{\Pi}_0$, $\mathbf{p}_0$, and the velocity $\boldsymbol{\xi} = (\xi_1, \xi_2)$ in the statement.

Notice first that $A_0\mathbf{e}_3 = \mathbf{e}_3$, hence by (5.40) we have that

$$\mathbf{p}_0 = M\xi_1 A_0^{-1}(-y, x, 0), \quad (5.41)$$

necessarily. Now by (5.39)

$$\boldsymbol{\Pi}_0 = \mathbb{I}_{\rm ref}(\xi_1 - \xi_2)\mathbf{e}_3 = I_3(\xi_1 - \xi_2)\mathbf{e}_3. \quad (5.42)$$

In order to handle (5.38) we note that $D\mathbf{B}(\mathbf{x})$ is given by the matrix whose components are

$$\frac{\partial B_x}{\partial x} = k\left(\frac{D(\mathbf{x})_+ - 3x^2}{D(\mathbf{x})_+^{5/2}} - \frac{D(\mathbf{x})_- - 3x^2}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_x}{\partial y} = k\left(\frac{-3xy}{D(\mathbf{x})_+^{5/2}} + \frac{3xy}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_x}{\partial z} = k\left(\frac{-3x(z-h)}{D(\mathbf{x})_+^{5/2}} + \frac{3x(z+h)}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_y}{\partial x} = k\left(\frac{-3xy}{D(\mathbf{x})_+^{5/2}} + \frac{3xy}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_y}{\partial y} = k\left(\frac{D(\mathbf{x})_+ - 3y^2}{D(\mathbf{x})_+^{5/2}} - \frac{D(\mathbf{x})_- - 3y^2}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_y}{\partial z} = k\left(\frac{-3y(z-h)}{D(\mathbf{x})_+^{5/2}} + \frac{3y(z+h)}{D(\mathbf{x})_-^{5/2}}\right),$$

$$\frac{\partial B_z}{\partial x} = k\left(\frac{-3x(z-h)}{D(\mathbf{x})_+^{5/2}} + \frac{3x(z+h)}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_z}{\partial y} = k\left(\frac{-3y(z-h)}{D(\mathbf{x})_+^{5/2}} + \frac{3y(z+h)}{D(\mathbf{x})_-^{5/2}}\right),$$
$$\frac{\partial B_z}{\partial z} = k\left(\frac{D(\mathbf{x})_+ - 3(z-h)^2}{D(\mathbf{x})_+^{5/2}} - \frac{D(\mathbf{x})_- - 3(z+h)^2}{D(\mathbf{x})_-^{5/2}}\right),$$

where $D(\mathbf{x})_+ = x^2+y^2+(z-h)^2$, $D(\mathbf{x})_- = x^2+y^2+(z+h)^2$ and $k = \dfrac{\mu_0 q}{4\pi}$. Consequently,

$$D\mathbf{B}(\mathbf{x}_0) = k\begin{pmatrix} 0 & 0 & \frac{6xh}{D(\mathbf{x}_0)^{5/2}} \\ 0 & 0 & \frac{6yh}{D(\mathbf{x}_0)^{5/2}} \\ \frac{6xh}{D(\mathbf{x}_0)^{5/2}} & \frac{6yh}{D(\mathbf{x}_0)^{5/2}} & 0 \end{pmatrix},$$

where $D(\mathbf{x}_0) = D(\mathbf{x}_0)_+ = D(\mathbf{x}_0)_-$. Hence

$$D\mathbf{B}\left(\mathbf{x}_0\right)^T\left(A_0\mathbf{e}_3\right) = D\mathbf{B}\left(\mathbf{x}_0\right)^T\mathbf{e}_3 = \frac{6kh}{D(\mathbf{x}_0)^{5/2}}\mathbf{x}_0. \tag{5.43}$$

Note additionally that by (5.41)

$$A_0\mathbf{p}_0 \times \mathbf{e}_3 = M\xi_1\mathbf{x}_0. \tag{5.44}$$

Then by equalities (5.43) and by (5.44), equation (5.38) holds whenever $\mathbf{x}_0 = \mathbf{0}$ or when $\mathbf{x}_0 \neq \mathbf{0}$ and $\xi_1^2 = -\dfrac{3h\mu_m q\mu_0}{2\pi M D(\mathbf{x}_0)^{5/2}}$; we note that in both situations, there are no restrictions on the second component of the velocity ξ_2. Finally, it can be readily verified that (5.37) always holds at the point $((A_0,\mathbf{x}_0),(\mathbf{\Pi}_0,\mathbf{p}_0))$ by using that $\mathbf{B}(\mathbf{x}_0) = -\dfrac{\mu_0 q h}{2\pi D(\mathbf{x}_0)^{3/2}}\mathbf{e}_3$.

(iii) Suppose that we are in the presence of a magnetic field $\mathbf{B}$ equivariant with respect to rotations around the OZ axis and that behaves as indicated in (2.7)–(2.9) with respect to the mirror transformation (2.6). Notice first that by (2.7) and (2.8)

$$B_x(x,y,0) = B_y(x,y,0) = 0 \tag{5.45}$$

and hence

$$\mathbf{B}(x,y,0) = B_z(x,y,0)\mathbf{e}_3. \tag{5.46}$$

Additionally, by (2.5), $B_z(x,y,0)$ is rotationally invariant with respect to rotations in the OXY plane, hence

$$B_z(x,y,0) = f(x^2+y^2), \text{ for some } f \in C^\infty(\mathbb{R}^2). \tag{5.47}$$

Conditions (3.7) and (3.6) show that if $A_0 = R^z_{\theta_0}$ and $\mathbf{x}_0 = (x,y,0)$, then $\mathbf{\Pi}_0 = I_3\left(\xi_1-\xi_2\right)\mathbf{e}_3$ and $\mathbf{p}_0 = M\xi_1 A_0^{-1}\left(-y,x,0\right)$ necessarily. If we use $\mathbf{z}_0 = ((A_0,\mathbf{x}_0),(\mathbf{\Pi}_0,\mathbf{p}_0))$ and (5.46) in the expression (3.4), it can be easily verified that this relation is automatically satisfied.

In order to study the expression (3.5), we take derivatives on both sides of (2.9) and obtain that

$$\partial_z B_z(x,y,z) = -\partial_z B_z(x,y,-z)$$

which shows that

$$\partial_z B_z(x,y,0) = 0. \tag{5.48}$$

Finally, by (5.46) and (5.48) the relation (3.5) amounts to

$$-\mu_m(\partial_x B_z, \partial_y B_z, 0) = M\xi_1^2\mathbf{x}_0.$$

By (5.47) this is equivalent to

$$-2\mu_m f'(x^2+y^2)\mathbf{x}_0 = M\xi_1^2\mathbf{x}_0,$$

which guarantees that (3.5) is satisfied provided that

$$\xi_1 = \pm\left(-\frac{2}{M}\mu_m f'(x^2+y^2)\right)^{1/2}, \tag{5.49}$$

as required. ■

5.5 Proof of Theorem 4.5

We will proceed by using Theorem 4.2 in order to determine the regions in parameter space for which the stability form (4.1) at the relative equilibria is definite, which in turn ensures $\mathbb{T}^2$–stability.

We start by denoting the augmented Hamiltonian as $H^{\boldsymbol{\xi}} := H - \boldsymbol{J}^{\boldsymbol{\xi}}$, for any $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathrm{Lie}\left(\mathbb{T}^2\right)$. Let $\mathbf{z} = ((A, \mathbf{x}), (\boldsymbol{\Pi}, \mathbf{p})) \in T^*(SE(3))$ expressed in body coordinates. As we saw in the proof of Proposition 3.1 (see Appendix 5.4), the partial derivatives of $H^{\boldsymbol{\xi}}$ are given by:

- $H^{\boldsymbol{\xi}}_A := D_A H^{\boldsymbol{\xi}}(\mathbf{z}) = \mu_m\left[\mathbf{B}(\mathbf{x}) \times A\mathbf{e}_3\right] + \xi_1\left[A\mathbf{p} \times (\mathbf{x} \times \mathbf{e}_3) - A\boldsymbol{\Pi} \times \mathbf{e}_3\right]$,
- $H^{\boldsymbol{\xi}}_{\mathbf{x}} := D_{\mathbf{x}} H^{\boldsymbol{\xi}}(\mathbf{z}) = -\mu_m D\mathbf{B}(\mathbf{x})^T(A\mathbf{e}_3) - \xi_1\left(A\mathbf{p} \times \mathbf{e}_3\right)$,
- $H^{\boldsymbol{\xi}}_{\boldsymbol{\Pi}} := D_{\boldsymbol{\Pi}} H^{\boldsymbol{\xi}}(\mathbf{z}) = \mathbb{I}_{\mathrm{ref}}^{-1}\boldsymbol{\Pi} + \xi_2\mathbf{e}_3 - \xi_1 A^{-1}\mathbf{e}_3$,
- $H^{\boldsymbol{\xi}}_{\mathbf{p}} := D_{\mathbf{p}} H^{\boldsymbol{\xi}}(\mathbf{z}) = \frac{1}{M}\mathbf{p} - \xi_1 A^{-1}\left(\mathbf{e}_3 \times \mathbf{x}\right)$.

In order to compute the Hessian of the augmented Hamiltonian, we write down the derivatives of its partial derivatives in the direction given by the vector $\mathbf{v} = \left.\frac{d}{dt}\right|_0 \Big((\exp t\widehat{\delta A}A, \mathbf{x} + t\boldsymbol{\delta}\mathbf{x}), (\boldsymbol{\Pi} + t\boldsymbol{\delta\Pi}, \mathbf{p} + t\boldsymbol{\delta}\mathbf{p})\Big)$. A straightforward computation yields:

- $\mathbf{d}H^{\boldsymbol{\xi}}_A(\mathbf{z}) \cdot \mathbf{v} = \mu_m\Big[(D\mathbf{B}(\mathbf{x})\boldsymbol{\delta}\mathbf{x}) \times A\mathbf{e}_3 + \mathbf{B}(\mathbf{x}) \times (\widehat{\delta A}A\mathbf{e}_3)\Big] + \xi_1\Big[(\widehat{\delta A}A\mathbf{p} + A\boldsymbol{\delta}\mathbf{p}) \times (\mathbf{x} \times \mathbf{e}_3) + A\mathbf{p} \times (\boldsymbol{\delta}\mathbf{x} \times \mathbf{e}_3) - \widehat{\delta A}A\boldsymbol{\Pi} \times \mathbf{e}_3 - (A\boldsymbol{\delta\Pi} \times \mathbf{e}_3)\Big]$,
- $\mathbf{d}H^{\boldsymbol{\xi}}_{\mathbf{x}}(\mathbf{z}) \cdot \mathbf{v} = -\mu_m\left(T_{\mathbf{x}}\mathbf{F}(\boldsymbol{\delta}\mathbf{x})\right)(A\mathbf{e}_3) - \mu_m\mathbf{F}(\mathbf{x})\left(\delta A \times A\mathbf{e}_3\right) - \xi_1(\widehat{\delta A}A\mathbf{p} \times \mathbf{e}_3 + A\boldsymbol{\delta}\mathbf{p} \times \mathbf{e}_3)$, where
$$\begin{aligned}\mathbf{F} : \mathbb{R}^3 &\longrightarrow M_{3\times 3}\\ \mathbf{x} &\longmapsto D\mathbf{B}(\mathbf{x})^T,\end{aligned}$$
- $\mathbf{d}H^{\boldsymbol{\xi}}_{\boldsymbol{\Pi}}(\mathbf{z}) \cdot \mathbf{v} = \mathbb{I}_{\mathrm{ref}}^{-1}\boldsymbol{\delta\Pi} + \xi_1 A^T\widehat{\delta A}\mathbf{e}_3$,
- $\mathbf{d}H^{\boldsymbol{\xi}}_{\mathbf{p}}(\mathbf{z}) \cdot \mathbf{v} = \dfrac{\boldsymbol{\delta}\mathbf{p}}{M} - \xi_1 A^T\widehat{\delta A}(\mathbf{x} \times \mathbf{e}_3) + \xi_1 A^T(\boldsymbol{\delta}\mathbf{x} \times \mathbf{e}_3)$.

Consequently, the matrix expression associated to $\mathbf{d}^2\left(H-\mathbf{J}^{\boldsymbol{\xi}}\right)(\mathbf{z})$ is given by:

$$\begin{pmatrix} -\mu_m\left[\widehat{\mathbf{B}(\mathbf{x})}\widehat{A\mathbf{e}_3}\right]+\xi_1\left[\widehat{\mathbf{x}\times\mathbf{e}_3}\widehat{A\mathbf{p}}-\widehat{\mathbf{e}_3}\widehat{A\mathbf{\Pi}}\right] & -\mu_m\widehat{A\mathbf{e}_3}\mathbf{F}(\mathbf{x})^T-\xi_1\widehat{A\mathbf{p}}\widehat{\mathbf{e}_3} & \xi_1\widehat{\mathbf{e}_3}A & -\xi_1\widehat{\mathbf{x}\times\mathbf{e}_3}A \\ \mu_m\left[\mathbf{F}(\mathbf{x})\widehat{A\mathbf{e}_3}\right]-\xi_1\widehat{\mathbf{e}_3}\widehat{A\mathbf{p}} & -\mu_m T_{\mathbf{x}}\mathbf{F}(\cdot)(A\mathbf{e}_3) & 0 & \xi_1\widehat{\mathbf{e}_3}A \\ -\xi_1A^T\widehat{\mathbf{e}_3} & 0 & \mathbb{I}_{\mathrm{ref}}^{-1} & 0 \\ \xi_1A^T\widehat{\mathbf{x}\times\mathbf{e}_3} & -\xi_1A^T\widehat{\mathbf{e}_3} & 0 & \dfrac{1}{M}\mathbb{I}_{id} \end{pmatrix} \tag{5.50}$$

We now compute the value of the Hessian (5.50) at the relative equilibria in the second and third parts of Proposition 3.1, that is, $\mathbf{z}_0=((A_0,\mathbf{x}_0),(\mathbf{\Pi}_0,\mathbf{p}_0))$ with $A_0=R_\theta^Z$, $\mathbf{x}_0=(x,y,0)^T$, $\mathbf{\Pi}_0=I_3\left(\xi_1^0-\xi_2\right)\mathbf{e}_3$, and $\mathbf{p}_0=M\xi_1^0A_0^{-1}(-y,x,0)^T$. We start by noticing that

$$T_{\mathbf{x}}\mathbf{F}(\cdot)(\mathbf{e}_3)=\mathrm{Hess}\left(B_z\right)(\mathbf{x}). \tag{5.51}$$

Indeed, for any $\boldsymbol{\delta}\mathbf{x}\in T_{\mathbf{x}}\mathbb{R}^3$

$$\begin{aligned} T_{\mathbf{x}}\mathbf{F}(\boldsymbol{\delta}\mathbf{x})(\mathbf{e}_3) &= \left.\frac{d}{dt}\right|_0 \mathbf{F}\left(\mathbf{x}+t\boldsymbol{\delta}\mathbf{x}\right)(\mathbf{e}_3)=\left.\frac{d}{dt}\right|_0 D\mathbf{B}\left(\mathbf{x}+t\boldsymbol{\delta}\mathbf{x}\right)^T\mathbf{e}_3 \\ &= \left.\frac{d}{dt}\right|_0 \begin{pmatrix} \dfrac{\partial B_z}{\partial x}\left(\mathbf{x}+t\boldsymbol{\delta}\mathbf{x}\right) \\ \dfrac{\partial B_z}{\partial y}\left(\mathbf{x}+t\boldsymbol{\delta}\mathbf{x}\right) \\ \dfrac{\partial B_z}{\partial z}\left(\mathbf{x}+t\boldsymbol{\delta}\mathbf{x}\right) \end{pmatrix} = \mathrm{Hess}\left(B_z\right)(\mathbf{x})\cdot\boldsymbol{\delta}\mathbf{x}. \end{aligned} \tag{5.52}$$

Therefore, the matrix expression associated to $\mathbf{d}^2\left(H-\mathbf{J}^{\boldsymbol{\xi}}\right)(\mathbf{z}_0)$ is given by:

$$\begin{pmatrix} -\mu_m\left[\widehat{\mathbf{B}(\mathbf{x}_0)}\widehat{\mathbf{e}_3}\right]+\xi_1\left[\widehat{\mathbf{x}_0\times\mathbf{e}_3}\widehat{\mathbf{p}_0}-\widehat{\mathbf{e}_3}\widehat{\mathbf{\Pi}_0}\right] & -\mu_m\widehat{\mathbf{e}_3}\mathbf{F}(\mathbf{x}_0)^T-\xi_1\widehat{\mathbf{p}_0}\widehat{\mathbf{e}_3} & \xi_1\widehat{\mathbf{e}_3} & -\xi_1(\widehat{\mathbf{x}_0\times\mathbf{e}_3}) \\ \mu_m\mathbf{F}(\mathbf{x}_0)\widehat{\mathbf{e}_3}-\xi_1\widehat{\mathbf{e}_3}\widehat{\mathbf{p}_0} & -\mu_m\mathrm{Hess}\left(B_z\right)(\mathbf{x}_0) & 0 & \xi_1\widehat{\mathbf{e}_3} \\ -\xi_1\widehat{\mathbf{e}_3} & 0 & \mathbb{I}_{\mathrm{ref}}^{-1} & 0 \\ \xi_1(\widehat{\mathbf{x}_0\times\mathbf{e}_3}) & -\xi_1\widehat{\mathbf{e}_3} & 0 & \dfrac{1}{M}\mathbb{I}_{id} \end{pmatrix}. \tag{5.53}$$

In order to construct stability forms for the regular and singular branches, we now determine stability spaces W to which we will restrict the Hessian (5.53).

A stability space for the regular branch ($r>0$). In this case, the kernel of the derivative of the momentum map is given by:

$$\begin{aligned} \ker T_{\mathbf{z}_0}\boldsymbol{J} &= \left\{v=\left((\widehat{\delta A},\boldsymbol{\delta}\mathbf{x}),(\boldsymbol{\delta\Pi},\boldsymbol{\delta}\mathbf{p})\right)\in T_{\mathbf{z}_0}\left(SE(3)\times\mathfrak{se}(3)^*\right)\mid T_{\mathbf{z}_0}\boldsymbol{J}\cdot v=0\right\} \\ &= \left\{v\in T_{\mathbf{z}_0}\left(SE(3)\times\mathfrak{se}(3)^*\right)\mid \boldsymbol{\delta\Pi}\cdot\mathbf{e}_3=0,\delta p_2=-M\xi_1^0\delta x_1\right\}, \end{aligned}$$

and using (5.29), the tangent space $\mathfrak{t}^2\cdot\mathbf{z}_0:=T_{\mathbf{z}_0}\left(\mathbb{T}^2\cdot\mathbf{z}_0\right)$ to the toral orbit that goes through the relative equilibrium $\mathbf{z}_0$ can be characterized as:

$$\begin{aligned} \mathfrak{t}^2\cdot\mathbf{z}_0 &= \left\{(\xi_1,\xi_2)_{T^*SE(3)}(\mathbf{z}_0)\mid\xi_1,\xi_2\in\mathbb{R}\right\}=\left\{\left((\widehat{\xi_1\mathbf{e}_3-\xi_2\mathbf{e}_3}),\xi_1\mathbf{e}_3\times\mathbf{x}_0,\mathbf{0},\xi_2\mathbf{e}_3\times\mathbf{p}_0\right)\mid\xi_1,\xi_2\in\mathbb{R}^2\right\} \\ &= \left\{\left(\left(\xi_1-\xi_2\right)\mathbf{e}_3,\xi_1r\mathbf{e}_2,\mathbf{0},-\xi_2Mr\xi_1^0\mathbf{e}_1\right)\mid\xi_1,\xi_2\in\mathbb{R}^2\right\}. \end{aligned}$$

Finally, it can be easily verified that the vector subspace $W \subset \ker T_{\mathbf{z}_0}\boldsymbol{J}$

$$W := \Big\{ (\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta\Pi}, \boldsymbol{\delta}\mathbf{p}) \mid \boldsymbol{\delta\Pi}\cdot\mathbf{e}_3 = 0, \boldsymbol{\delta}\mathbf{x}\cdot\mathbf{p}_0 = 0, \boldsymbol{\delta}\mathbf{p}\cdot\mathbf{x}_0 = 0, \delta p_2 = -M\xi_1^0 \delta x_1 \Big\}$$

is such that

$$\mathrm{Ker} T_{\mathbf{z}_0}\boldsymbol{J} = W \oplus \mathfrak{t}^2 \cdot \mathbf{z}_0, \tag{5.54}$$

and hence constitutes a stability space. Moreover, let $\mathbf{u}_1 = \big(\mathbf{0}, \mathbf{e}_1, \mathbf{0}, -M\xi_1^0\mathbf{e}_2\big)$, $\mathbf{u}_2 = (\mathbf{e}_3, \mathbf{0}, \mathbf{0}, \mathbf{0})$, $\mathbf{u}_3 = (\mathbf{0}, \mathbf{0}, \mathbf{e}_1, \mathbf{0})$, $\mathbf{u}_4 = (\mathbf{0}, \mathbf{0}, \mathbf{e}_2, \mathbf{0})$, $\mathbf{u}_5 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_3)$, $\mathbf{u}_6 = (\mathbf{0}, \mathbf{e}_3, \mathbf{0}, \mathbf{0})$, $\mathbf{u}_7 = (\mathbf{e}_2, \mathbf{0}, \mathbf{0}, \mathbf{0})$, and $\mathbf{u}_8 = (\mathbf{e}_1, \mathbf{0}, \mathbf{0}, \mathbf{0})$. It can be checked that

$$W = \mathrm{span}\Big\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8\Big\}. \tag{5.55}$$

The set $\mathcal{B} = \Big\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8\Big\}$ will be used as a basis of the stability space in order to obtain matrix expressions for the stability form $\mathbf{d}^2\left(H - \boldsymbol{J}^{(\xi_1^0, \xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ corresponding to each part of Theorem 4.5.

A stability space for the singular branch ($r = 0$). Consider now the relative equlibrium $z_0 = ((A_0, \mathbf{x}_0), (\boldsymbol{\Pi}_0, \mathbf{p}_0))$ with $A_0 = R^Z_{\theta_0}$, $\mathbf{x}_0 = (0,0,0)$, $\boldsymbol{\Pi}_0 = I_3(\xi_1 - \xi_2)\,\mathbf{e}_3$, and $\mathbf{p}_0 = \mathbf{0}$, $\xi_1, \xi_2 \in \mathbb{R}$. In this case, the matrix expression (5.50) associated to $\mathbf{d}^2\left(H - \mathbf{J}^{\boldsymbol{\xi}}\right)(\mathbf{z})$ is given by:

$$\begin{pmatrix} -\mu_m\widehat{\mathbf{B}(\mathbf{x}_0)}\widehat{\mathbf{e}_3} - \xi_1\widehat{\mathbf{e}_3}\widehat{\boldsymbol{\Pi}_\mathbf{0}} & 0 & \xi_1\widehat{\mathbf{e}_3} & 0 \\ 0 & -\mu_m \mathrm{Hess}\,(B_z)(\mathbf{x}_0) & 0 & \xi_1\widehat{\mathbf{e}_3} \\ -\xi_1\widehat{\mathbf{e}_3} & 0 & \mathbb{I}_{\mathrm{ref}}^{-1} & 0 \\ 0 & -\xi_1\widehat{\mathbf{e}_3} & 0 & \dfrac{1}{M}\mathbb{I}_{id} \end{pmatrix}. \tag{5.56}$$

These relative equilibria lay on the singular isotropy type manifold (2.18) and hence by the Bifurcation Lemma (see [OR04, Proposition 4.5.12]), the kernel of the derivative of the momentum map is necessarily of dimension eleven at those points. Indeed, it can be checked that:

$$\ker T_{\mathbf{z}_0}\boldsymbol{J} = \left\{ v \in T_{\mathbf{z}_0}\left(SE(3)\times\mathfrak{se}(3)^*\right) \mid \boldsymbol{\delta\Pi}\cdot\mathbf{e}_3 = 0 \right\},$$

and using (5.29), the tangent space $\mathfrak{t}^2\cdot\mathbf{z}_0 := T_{\mathbf{z}_0}\left(\mathbb{T}^2\cdot\mathbf{z}_0\right)$ to the toral orbit that goes through the singular relative equilibrium $\mathbf{z}_0$ can be characterized as:

$$\mathfrak{t}^2\cdot\mathbf{z}_0 = \left\{((\xi_1 - \xi_2)\,\mathbf{e}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}) \mid \xi_1, \xi_2 \in \mathbb{R}^2\right\} = \mathrm{span}\{(\mathbf{e}_3, \mathbf{0}, \mathbf{0}, \mathbf{0})\}.$$

Finally, it can be easily verified that the vector subspace $W \subset \ker T_{\mathbf{z}_0}\boldsymbol{J}$ given by

$$W = \mathrm{span}\Big\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_9, \mathbf{u}_{10}\Big\}, \tag{5.57}$$

with $\mathbf{u}_1 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_3)$, $\mathbf{u}_2 = (\mathbf{0}, \mathbf{e}_3, \mathbf{0}, \mathbf{0})$, $\mathbf{u}_3 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_2)$, $\mathbf{u}_4 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_1)$, $\mathbf{u}_5 = (\mathbf{0}, \mathbf{0}, \mathbf{e}_2, \mathbf{0})$, $\mathbf{u}_6 = (\mathbf{0}, \mathbf{0}, \mathbf{e}_1, \mathbf{0})$, $\mathbf{u}_7 = (\mathbf{0}, \mathbf{e}_2, \mathbf{0}, \mathbf{0})$, $\mathbf{u}_8 = (\mathbf{0}, \mathbf{e}_1, \mathbf{0}, \mathbf{0})$, $\mathbf{u}_9 = (\mathbf{e}_2, \mathbf{0}, \mathbf{0}, \mathbf{0})$, and $\mathbf{u}_{10} = (\mathbf{e}_1, \mathbf{0}, \mathbf{0}, \mathbf{0})$ is a $H-$invariant stability space, that is,

$$\mathrm{Ker} T_{\mathbf{z}_0}\boldsymbol{J} = W \oplus \mathfrak{t}^2 \cdot \mathbf{z}_0, \tag{5.58}$$

and hence constitutes a stability space. We recall that $H := \left\{\left(e^{i\theta}, e^{i\theta}\right) \mid e^{i\theta} \in S^1\right\} = \mathbb{T}^2_{\mathbf{z}_0}$ is the isotropy subgroup of the relative equilibrium $\mathbf{z}_0$. We will use the set $\mathcal{B} = \Big\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_9, \mathbf{u}_{10}\Big\}$

as a basis of the stability space in order to obtain a matrix expression for the stability form $\mathbf{d}^2\left(H - \boldsymbol{J}^{(\xi_1,\xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ for the parts **(i)** and **(ii)** of Theorem 4.5.

Proof of part (i) of the theorem.

Stability study for the regular branch. We start by noting that the stability of a relative equilibrium can be determined by using any of the the points that constitute its trajectory in phase space. Hence we can, without loss of generality, use the relative equilibrium point $\mathbf{z}_0$ of the form $\mathbf{z}_0 = \left((\mathbb{I}_{id}, r\mathbf{e}_1), (I_3\left(\xi_1^0 - \xi_2\right)\mathbf{e}_3, Mr\xi_1^0\mathbf{e_2})\right)$. We recall that the regular relative equilibria are those for which $r > 0$ and

$$\xi_1^0 = \pm\left(-\frac{3h\mu_m q\mu_0}{2\pi M D(\mathbf{x}_0)^{5/2}}\right)^{1/2}.$$

We now provide the expression of $\operatorname{Hess}(B_z)(\mathbf{x}_0)$ using the same notation as in (5.43) and conclude that

$$\operatorname{Hess}(B_z)(\mathbf{x}_0) = \frac{6kh}{D(\mathbf{x}_0)^{7/2}}\begin{pmatrix} D(\mathbf{x}_0) - 5x^2 & -5xy & 0 \\ -5xy & D(\mathbf{x}_0) - 5y^2 & 0 \\ 0 & 0 & 3D(\mathbf{x}_0) - 5h^2 \end{pmatrix}.$$

By (5.53) and (5.55) we obtain that the stability form $\mathbf{d}^2\left(h - \boldsymbol{J}^{(\xi_1^0,\xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ can be written as:

$$\begin{pmatrix}
M\xi_1^0\dfrac{4h^2-r^2}{r^2+h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & M{\xi_1^0}^2r^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{1}{I_1} & 0 & 0 & 0 & \xi_1^0 & 0 \\
0 & 0 & 0 & \dfrac{1}{I_1} & 0 & 0 & 0 & -\xi_1^0 \\
0 & 0 & 0 & 0 & \dfrac{1}{M} & 0 & 0 & \xi_1^0 r \\
0 & 0 & 0 & 0 & 0 & M{\xi_1^0}^2\dfrac{3r^2-2h^2}{r^2+h^2} & M{\xi_1^0}^2 r & 0 \\
0 & 0 & \xi_1^0 & 0 & 0 & M{\xi_1^0}^2 r & \dfrac{1}{3}M{\xi_1^0}^2(r^2+h^2)+\xi_1^0\Pi_0 & 0 \\
0 & 0 & 0 & -\xi_1^0 & \xi_1^0 r & 0 & 0 & \dfrac{1}{3}M{\xi_1^0}^2(4r^2+h^2)+\xi_1^0\Pi_0
\end{pmatrix}, \tag{5.59}$$

where $\Pi_0 = I_3\left(\xi_1^0 - \xi_2\right)$. Notice that this matrix is block diagonal and exhibits two blocks of size two and six. The positivity of the block of size two requires that $M{\xi_1^0}^2 r^2 > 0$ and $4h^2 - r^2 > 0$. The first inequality is always satisfied when $\mu_m q < 0$ and the second one amounts to

$$\frac{r^2}{h^2} < 4, \tag{5.60}$$

which yields the right hand side inequality in (4.2). We now study the positivity of the lower six dimensional block of the stability form. Given that by Sylvester's Law of Inertia the signature of a diagonalizable matrix is invariant with respect to conjugation by invertible matrices, it can hence be read out of the pivots of the matrix obtained by performing Gaussian elimination on this block. Indeed, these pivots are

$$\frac{1}{I_1},\ \frac{1}{I_1},\ \frac{1}{M},\ p_1,\ p_2,\ p_3, \tag{5.61}$$

where

$$\begin{aligned} p_1 &:= M{\xi_1^0}^2\frac{3r^2-2h^2}{r^2+h^2}, \\ p_2 &:= -I_3\xi_1^0\xi_2-{\xi_1^0}^2\left(\frac{2}{3}M\frac{(r^2+h^2)h^2}{3r^2-2h^2}+(I_1-I_3)\right), \\ p_3 &:= -I_3\xi_1^0\xi_2-{\xi_1^0}^2\left(-\frac{1}{3}M(r^2+h^2)+(I_1-I_3)\right). \end{aligned}$$

The first three are automatically positive. The positivity of p_1 is equivalent to

$$\frac{2}{3}<\frac{r^2}{h^2},$$

which yields the left hand side inequality in (4.2) Finally, we study the positivity of the last two pivots p_2 and p_3. The simultaneous positivity of p_2 and p_3 is equivalent to $\min\{p_2,p_3\}>0$. It is easy to check that $\min\{p_2,p_3\}=p_2$, since the condition

$$p_3-p_2=M{\xi_1^0}^2r^2\frac{r^2+h^2}{3r^2-2h^2}>0 \tag{5.62}$$

is always satisfied due to (5.5) and the condition $\mu_m q<0$. Regarding the positivity of p_2 there are two possible cases:
1) $\xi_1^0>0$, then

$$I_3(\xi_1^0-\xi_2)>\xi_1^0\left(I_1+\frac{2}{3}M\frac{(r^2+h^2)h^2}{3r^2-2h^2}\right),$$

2) $\xi_1^0<0$, then

$$I_3(\xi_1^0-\xi_2)<\xi_1^0\left(I_1+\frac{2}{3}M\frac{(r^2+h^2)h^2}{3r^2-2h^2}\right),$$

and hence the positivity of p_2 can be summarized as

$$\operatorname{sign}(\xi_1^0)I_3\xi_2<-|\xi_1^0|\left(I_1-I_3+\frac{2}{3}M\frac{(r^2+h^2)h^2}{3r^2-2h^2}\right),$$

which coincides with (4.3), as required.

Stability study for the singular branch. We first notice that the matrix expression associated to $\mathbf{d}^2\left(H-\mathbf{J}^\xi\right)(\mathbf{z})$ is given by (5.56), where

$$\mathbf{B}(\mathbf{x}_0)=-\frac{\mu_0 q}{2\pi h^2}\mathbf{e}_3 \text{ and } \operatorname{Hess}(B_z)(\mathbf{x}_0)=-\frac{3\mu_0 q\mu_m}{2\pi h^4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \tag{5.63}$$

Consequently, the pivots obtained by Gaussian elimination in the matrix expression of the stability form $\mathbf{d}^2\left(H-\boldsymbol{J}^{(\xi_1,\xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ are

$$\frac{1}{M},\frac{3\mu_0 q\mu_m}{\pi h^4},\frac{1}{M},\frac{1}{M},\frac{1}{I_1},\frac{1}{I_1},p_1,p_1,p_2,p_2, \tag{5.64}$$

where

$$p_1=-\frac{3}{2}\frac{\mu_0 q\mu_m}{\pi h^4}-M\xi_1^2,\quad p_2=-\frac{\mu_0 q\mu_m}{2\pi h^2}-\xi_1(\xi_1 I_1-\Pi_0),\quad \text{and}\quad \Pi_0=I_3(\xi_1-\xi_2).$$

The formal instability of the singular branch is caused by the fact that the pivots in (5.64) cannot simultaneously have all the same sign. Indeed, $1/M$ and $1/I_1$ are always positive which forces $3\mu_0 q\mu_m/\pi h^4 > 0$. This is in turn incompatible with $p_1 > 0$ because that would require $M\xi_1^2 < 0$, which is not possible.

Proof of part (ii) of the theorem.

Stability study for the regular branch. In order to prove the second part of Theorem 4.5, we follow the same pattern that we used above. Let $f \in C^\infty(\mathbb{R}^2)$ be the function such that

$$B_z(x, y, z) := f(x^2 + y^2, z) \tag{5.65}$$

and $f_0 := f(x^2 + y^2, 0)$. Additionally,

$$f_1' := \left.\frac{\partial f(v, z)}{\partial v}\right|_{v=x^2+y^2, z=0}, \quad f_1'' := \left.\frac{\partial^2 f(v, z)}{\partial v^2}\right|_{v=x^2+y^2, z=0}, \quad f_2'' := \left.\frac{\partial^2 f(v, z)}{\partial z^2}\right|_{v=x^2+y^2, z=0},$$

and we recall that $\xi_1^0 = \pm\left(-\frac{2}{M}\mu_m f_1'\right)^{1/2}$. We now compute the components of the matrix $D\mathbf{B}(\mathbf{x}_0)$. Using the equations (2.7)–(2.9) and (5.65) we obtain

$$\left.\frac{\partial B_x}{\partial x}\right|_{\mathbf{x}_0} = \left.\frac{\partial B_x}{\partial y}\right|_{\mathbf{x}_0} = 0, \quad \left.\frac{\partial B_y}{\partial x}\right|_{\mathbf{x}_0} = \left.\frac{\partial B_y}{\partial y}\right|_{\mathbf{x}_0} = 0, \quad \left.\frac{\partial B_z}{\partial z}\right|_{\mathbf{x}_0} = 0, \quad \left.\frac{\partial B_z}{\partial x}\right|_{\mathbf{x}_0} = 2xf_1', \quad \left.\frac{\partial B_z}{\partial y}\right|_{\mathbf{x}_0} = 2yf_1'.$$

In order to determine the remaining two components in $D\mathbf{B}(\mathbf{x}_0)$, we use the Ampère-Maxwell equation $\nabla \times \mathbf{B} = 0$ in the absence of additional currents and time-varying electric fields in the region where the body motion takes place. Indeed, $\nabla \times \mathbf{B} = 0$ implies that

$$\left.\frac{\partial B_x}{\partial z}\right|_{\mathbf{x}_0} = \left.\frac{\partial B_z}{\partial x}\right|_{\mathbf{x}_0} = 2xf_1', \quad \left.\frac{\partial B_y}{\partial z}\right|_{\mathbf{x}_0} = \left.\frac{\partial B_z}{\partial y}\right|_{\mathbf{x}_0} = 2yf_1',$$

and hence,

$$D\mathbf{B}(\mathbf{x}_0) = \begin{pmatrix} 0 & 0 & 2xf_1' \\ 0 & 0 & 2yf_1' \\ 2xf_1' & 2yf_1' & 0 \end{pmatrix}.$$

By expression (5.51)

$$T_{\mathbf{x}_0}\mathbf{F}(\cdot)(\mathbf{e}_3) = \mathrm{Hess}\,(B_z)\,(\mathbf{x}_0) = \begin{pmatrix} 2f_1' + 4x^2 f_1'' & 4xyf_1'' & 0 \\ 4xyf_1'' & 2f_1' + 4y^2 f_1'' & 0 \\ 0 & 0 & f_2'' \end{pmatrix}. \tag{5.66}$$

Using the same argument as in the proof of part **(i)** we use, without loss of generality, the relative equilibrium point $\mathbf{z}_0$ of the form $\mathbf{z}_0 = ((\mathbb{I}_{id}, r\mathbf{e}_1), (I_3\left(\xi_1^0 - \xi_2\right)\mathbf{e}_3, Mr\xi_1^0\mathbf{e_2}))$, where $r > 0$. The matrix expression of $\mathbf{d}^2\left(H - \boldsymbol{J}^{(\xi_1^0, \xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ is:

$$\left(\begin{array}{cccccccc}
-2\mu_m(f_1'+2r^2f_1'')+3{\xi_1^0}^2M & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & Mr^2{\xi_1^0}^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & \xi_1^0 & 0\\
0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & -\xi_1^0\\
0 & 0 & 0 & 0 & \frac{1}{M} & 0 & 0 & \xi_1^0 r\\
0 & 0 & 0 & 0 & 0 & -\mu_m f_2'' & -2\mu_m r f_1' & 0\\
0 & 0 & \xi_1^0 & 0 & 0 & -2\mu_m r f_1' & \mu_m f_0+\xi_1^0\Pi_0 & 0\\
0 & 0 & 0 & -\xi_1^0 & \xi_1^0 r & 0 & 0 & \mu_m f_0+\xi_1^0(Mr^2\xi_1^0+\Pi_0)
\end{array}\right), \tag{5.67}$$

where $\Pi_0 = I_3\left(\xi_1^0 - \xi_2\right)$. Notice that this matrix is block diagonal and exhibits two blocks of size two and six. The positivity of the block of size two requires that $\mu_m f_1' < 0$ and $\mu_m(2f_1' + r^2 f_1'') < 0$ which coincide with (4.4) and (4.5). We now study the positivity of the lower six dimensional block of the stability form. As we did in the proof of part **(i)**, we will read the signature of this block out of its pivots, which are $\frac{1}{I_1}$, $\frac{1}{I_1}$, $\frac{1}{M}$, $-\mu_m f_2''$, $\xi_1^0\left(\Pi_0 - \xi_1^0 I_1\right) + \mu_m\left(f_0 + 4r^2\frac{{f_1'}^2}{f_2''}\right)$, and $\mu_m f_0 + \xi_1^0\Pi_0 - {\xi_1^0}^2 I_1$. The first three are automatically positive. The positivity of the fourth requires

$$\mu_m f_2'' < 0, \tag{5.68}$$

which corresponds to the inequality (4.6) in the statement. Finally, we study the positivity of the last two pivots. Let

$$p_1 := \xi_1^0\left(\Pi_0 - \xi_1^0 I_1\right) + \mu_m\left(f_0 + 4r^2\frac{{f_1'}^2}{f_2''}\right) \tag{5.69}$$

and

$$p_2 := \mu_m f_0 + \xi_1^0\Pi_0 - {\xi_1^0}^2 I_1. \tag{5.70}$$

The simultaneous positivity of p_1 and p_2 is equivalent to $\min\{p_1, p_2\} > 0$. It is easy to check that $\min\{p_1, p_2\} = p_1$, since the condition

$$p_2 - p_1 = -4\mu_m r^2\frac{{f_1'}^2}{f_2''} > 0 \tag{5.71}$$

is satisfied due to (5.68). The positivity of p_1 can be summarized as

$$\operatorname{sign}(\xi_1^0)I_3\xi_2 < -|\xi_1^0|\left((I_1 - I_3) + \frac{1}{2}M\left(\frac{f_0}{f_1'} + 4r^2\frac{f_1'}{f_2''}\right)\right),$$

which coincides with (4.7), as required.

Stability study for the singular branch. The matrix expression associated to $\mathbf{d}^2\left(H - \mathbf{J}^\xi\right)(\mathbf{z})$ is given by (5.56), where in this case

$$\mathbf{B}(\mathbf{x}_0) = B_z(\mathbf{x}_0)\mathbf{e}_3 = f_0\mathbf{e}_3 \text{ and } \operatorname{Hess}(B_z)(\mathbf{x}_0) = \begin{pmatrix} 2f_1' & 0 & 0\\ 0 & 2f_1' & 0\\ 0 & 0 & f_2''\end{pmatrix}. \tag{5.72}$$

The pivots obtained by Gaussian elimination in the matrix expression of the stability form $\mathbf{d}^2\left(H - \boldsymbol{J}^{(\xi_1,\xi_2)}\right)(\mathbf{z}_0)\Big|_{W\times W}$ are

$$p_1, p_2, p_1, p_1, p_3, p_3, p_4, p_4, p_5, p_5,$$

where

$$p_1 = \frac{1}{M},\ \ p_2 = -\mu_m f_2'',\ \ p_3 = \frac{1}{I_1},\ \ p_4 = -2\mu_m f_1' - M\xi_1^2,\ \ p_5 = \mu_m f_0 - \xi_1(\xi_1 I_1 - \Pi_0),\ \text{ and }\ \Pi_0 = I_3(\xi_1 - \xi_2).$$

The pivots p_1 and p_3 are automatically positive. The positivity of the pivots p_2, p_4, and p_5 requires that:

$$\mu_m f_2'' < 0, \tag{5.73}$$
$$\mu_m f_1' < 0, \tag{5.74}$$
$$\xi_1^2 < -\frac{2}{M}\mu_m f_1', \tag{5.75}$$
$$\operatorname{sign}(\xi_1)\Pi_0 > \frac{I_1\xi_1^2 - \mu_m f_0}{|\xi_1|}, \tag{5.76}$$

which yields the conditions (4.9)–(4.11). We now derive the optimal stability condition (4.12). Let $g(\xi_1) := \left(I_1\xi_1^2 - \mu_m f_0\right)/\xi_1$ in (4.11). It is easy to verify that the function $g(\xi_1)$ has a minimum at $\widehat{\xi_1^+} = \sqrt{-\mu_m f_0/I_1}$ and a maximum at $\widehat{\xi_1^-} = -\sqrt{-\mu_m f_0/I_1}$ provided that $\mu_m f_0 < 0$. Since the condition (4.10) has to be satisfied, then $f_0/f_1' < 2I_1/M$ also needs to hold. In that case, the choices $\widehat{\xi_1^\pm} = \pm\sqrt{-\mu_m f_0/I_1}$ and the inequalities

$$\Pi_0 > \min_{\xi_1\in\mathbb{R}^+}\{g(\xi_1)\} = g\left(\widehat{\xi_1^+}\right) = 2\sqrt{-\mu_m f_0 I_1}, \qquad \Pi_0 < \max_{\xi_1\in\mathbb{R}^-}\{g(\xi_1)\} = g\left(\widehat{\xi_1^-}\right) = -2\sqrt{-\mu_m f_0 I_1}$$

determine the largest possible stability region in the Π_0 (and consequently the ξ_2) variable, as required in (4.12).

We now prove the claim in Remark 4.10 about the stability conditions (4.8)–(4.11) being also valid for magnetic fields that do not satisfy $\nabla\times\mathbf{B} = 0$ as long as the conditions (4.13) are satisfied. It suffices to notice that $\mathbf{F}(\mathbf{x}_0)\widehat{\mathbf{e}_3} = \mathbf{0}$ by (4.13); this equality substituted in (5.53) yields the same second derivative (5.56) obtained under the hypothesis $\nabla\times\mathbf{B} = 0$. The stability conditions implied by the two hypotheses hence coincide. ■

5.6 Proofs of Propositions 4.12 and 4.13

Proof of Proposition 4.12

(i) It is a consequence of the Witt-Artin decomposition (see for example [OR04, Theorem 7.1.1]).

(ii) It is a consequence of the fact that the symplectic slice introduced by Marle [Mar84, Mar85], Guillemin, and Sternberg [GS84] can be constructed by Riemannian exponentiation of a symplectic tube. Since we need this construction in the proof of the following parts of the proposition, we briefly recall it using the notation in Chapter 7 of [OR04].

The first step is the splitting of the Lie algebra $\mathfrak{g}$ of G into three parts. The first summand is $\mathfrak{g}_\mu := \mathrm{Lie}\,(G_\mu)$. The equivariance of the momentum map $\mathbf{J}$ implies that $G_m \subset G_\mu$ and hence $\mathfrak{g}_m \subset \mathfrak{g}_\mu$. Hence we can fix an Ad_{G_m}-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ (always available by the compactness of G_m) and write

$$\mathfrak{g}_\mu = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q}, \tag{5.77}$$

where $\mathfrak{m}$ is the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathfrak{g}_m$ in $\mathfrak{g}_\mu$ and $\mathfrak{q}$ is the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. The splittings in (5.77) induce similar ones on the duals

$$\mathfrak{g}_\mu^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*. \tag{5.78}$$

Each of the spaces in this decomposition should be understood as the set of covectors in $\mathfrak{g}^*$ that can be written as $\langle\xi,\cdot\rangle$, with ξ in the corresponding subspace. For example, $\mathfrak{q}^* = \{\langle\xi,\cdot\rangle \mid \xi\in\mathfrak{q}\}$.

The second ingredient in the construction of the symplectic tube comes from noting that the compact (by the properness of the action) isotropy subgroup G_m acts linearly and canonically on (W,ω_W) with momentum map $\mathbf{J}_W : W \to \mathfrak{g}_m^*$ given by

$$\langle \mathbf{J}_W(w),\eta\rangle = \frac{1}{2}\omega_W\left(\eta_W(w),w\right), \qquad \eta\in\mathfrak{g}_m.$$

It can be shown [OR04, Proposition 7.2.2] that there exist G_m–invariant neighborhoods $\mathfrak{m}_r^*$ and W_r of the origin in $\mathfrak{m}^*$ and W, respectively, such that the twisted product $Y_r := G\times_{G_m}(\mathfrak{m}_r^*\times W_r)$ is endowed with a natural symplectic form ω_{Y_r} whose expression can be found in (7.2.2) of [OR04]. The Lie group G acts canonically on (Y_r,ω_{Y_r}) by $g\cdot[h,\eta,w]=[gh,\eta,w]$, for any $g\in G$ and $[h,\eta,w]\in Y_r$, and has a momentum map $\mathbf{J}_{Y_r}: Y_r\to\mathfrak{g}^*$ associated given by the so called Marle–Guillemin–Sternberg normal form:

$$\mathbf{J}_{Y_r}\left([g,\eta,w]\right) = \mathrm{Ad}^*_{g^{-1}}\left(\mu+\eta+\mathbf{J}_W(w)\right), \qquad [g,\eta,w]\in Y_r.$$

The G–symplectic manifold (Y_r,ω_{Y_r}) is called a symplectic tube of (M,ω) at the point m. This denomination is justified by the Symplectic Slice Theorem [Mar84, Mar85, GS84] that proves the existence of a G–equivariant symplectomorphism $\phi: U\to Y_r$ between a G–invariant neighborhood U of m in M and Y_r satisfying $\phi(m)=[e,0,0]$. The symplectic slice S in the statement of the proposition is obtained [OR04, Theorem 7.4.1] as $S=\phi^{-1}(S_{Y_r})$, where $S_{Y_r} := \{[e,0,w]\mid w\in W_r\}$ and, more explicitly, as $S=\{\mathrm{Exp}_m(w)\mid w\in W_r\}$, with Exp_m the Riemannian exponential associated to a G_m–invariant metric. The identity $T_mS=W$ is a consequence of the fact that $T_0\mathrm{Exp}_m=\mathrm{Id}$.

(iii) Since $m\in M$ is a relative equilibrium, we have $\mathbf{d}\left(H-\mathbf{J}^\xi\right)(m)=0$. This implies that $\mathbf{d}H_S^\xi(m) = \mathbf{d}\left(H-\mathbf{J}^\xi\right)(m)\big|_{T_mS}=0$ and hence $X_{H_S^\xi}(m)=0$.

(iv) This statement is a consequence of the combination of **(ii)** and **(iii)** with the following lemma.

Lemma 5.2 *Let (M,ω) be a symplectic manifold, $H\in C^\infty(M)$, and X_H the corresponding Hamiltonian vector field. Suppose that $m_0\in M$ is an equilibrium point of X_H, that is $X_H(m_0)=0$ and consequently $\mathbf{d}H(m_0)=0$. Then, the linearization X' of X_H at m_0 is a Hamiltonian vector field on the symplectic vector space $(T_{m_0}M,\omega(m_0))$ with Hamiltonian function $Q\in C^\infty(T_{m_0}M)$ given by*

$$Q(v)=\frac{1}{2}\mathbf{d}^2H(m_0)(v,v). \tag{5.79}$$

Proof of the Lemma. Note $V=T_{m_0}M$ and $\omega_V=\omega(m_0)$. Let $v,w\in V$ arbitrary and let $\{c(s)|s\in\mathbb{R}\}$ be a curve such that $v=\frac{d}{ds}\big|_{s=0}c(s)$. Then if F_t is the flow of X_h, we write

$$\omega_V\left(X'(v),w\right) = \left.\frac{d}{dt}\right|_{t=0}\omega(m_0)\left(T_{m_0}F_t\cdot v,w\right) = \left.\frac{d}{dt}\right|_{t=0}\omega(m_0)\left(\left.\frac{d}{ds}\right|_{s=0}F_t(c(s)),w\right), \quad v,w\in V. \tag{5.80}$$

We now take a Darboux chart (U,ϕ) [AM78, page 75] around the point m_0. Recall that in Darboux coordinates, the symplectic form ω_U is constant. Additionally if $\phi: U\longrightarrow\phi(U)\subset\mathbb{R}^n$, let $u\in\mathbb{R}^n$ be such that $T_{m_0}\phi \cdot w = (\phi(m_0), u) \in \phi(U) \times \mathbb{R}^n = T(\phi(U))$. Now, since $\phi^*\omega_U = \omega|_U$, then (5.80) can be written as

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \omega(m_0)\left(\left.\frac{d}{ds}\right|_{s=0} F_t(c(s)), w\right) &= \left.\frac{d}{dt}\right|_{t=0} \omega_U\left(\left.\frac{d}{ds}\right|_{s=0} \phi \cdot F_t(c(s)), T_{m_0}\phi \cdot w\right) \\
&= \omega_U\left(\left.\frac{d}{ds}\right|_{s=0} T_{c(s)}\phi \cdot X_H(c(s)), (\phi(m_0), u)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \omega_U\left(T_{c(s)}\phi \cdot X_{H\circ\phi^{-1}\circ\phi}(c(s)), (\phi(c(s)), u)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \omega_U\left(X_{H\circ\phi^{-1}}(\phi(c(s))), (\phi(c(s)), u)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \mathbf{d}\left(H\circ\phi^{-1}\right)(\phi(c(s)))\cdot(\phi(c(s)), T_{m_0}\phi\cdot w) \\
&= \mathbf{d^2}\left(H\circ\phi^{-1}\right)(\phi(m_0))\left((\phi(m_0), T_{m_0}\phi\cdot v), (\phi(m_0), T_{m_0}\phi\cdot w)\right) \\
&= \mathbf{d^2}H(m_0)(v,w) = \mathbf{d}Q(v)\cdot w.
\end{aligned}$$

Consequently, $\mathbf{i}_{X'}\omega_V = \mathbf{d}Q$, as required. ▼

(v) The hypothesis $T_m(G_\mu \cdot m) = T_m(G\cdot m)$ implies that $\mathfrak{q}\cdot m := \{\xi_M(m) \mid \xi \in \mathfrak{q}\}$; this fact and the construction of the Witt–Artin decomposition (see for example the expression (7.1.11) in [OR04]) ensure that (4.15) holds. In order to prove (4.16), notice that for any $w_1, w_2 \in W$

$$\begin{aligned}
\omega_W(X_Q(w_1), w_2) = \mathbf{d}Q(w_1)\cdot w_2 = \mathbf{d}^2 H^\xi(m)(w_1, w_2) &= \omega(m)(X'_{H^\xi}(w_1), w_2) \\
&= \omega(m)\left(\mathbb{P}_W X'_{H^\xi}(w_1), w_2\right) + \omega(m)\left((\mathbb{I}-\mathbb{P}_W)X'_{H^\xi}(w_1), w_2\right) \\
&= \omega(m)\left(\mathbb{P}_W X'_{H^\xi}(w_1), w_2\right) = \omega_W\left(\mathbb{P}_W X'_{H^\xi}(w_1), w_2\right),
\end{aligned}$$

where we used that $(\mathbb{I}-\mathbb{P}_W)X'_{H^\xi}(w_1) \in W^\omega$ and hence $\omega(m)\left((\mathbb{I}-\mathbb{P}_W)X'_{H^\xi}(w_1), w_2\right) = 0$. Since $w_1, w_2 \in W$ are arbitrary, the equality $\omega_W(X_Q(w_1), w_2) = \omega_W\left(\mathbb{P}_W X'_{H^\xi}(w_1), w_2\right)$ implies that

$$X_Q(w) = \mathbb{P}_W X'_{H^\xi}(w), \quad \forall w \in W,$$

which is equivalent to (4.16).

(vi) Given the local and group invariant character of this statement, we will prove this statement using the so called reconstruction differential equations [Ort98, RWL02, OR04] that determine the Hamiltonian vector field associated to a G–invariant Hamiltonian $h \in C^\infty(Y_r)$ in the symplectic tube Y_r. Consider first $\pi : G \times \mathfrak{m}_r^* \times W_r \longrightarrow G \times_{G_m} (\mathfrak{m}_r^* \times W_r) = Y_r$ the orbit projection; the G–invariance of H implies that the composition $H\circ\pi$ can be understood as a G_m–invariant function on $G\times\mathfrak{m}_r^*\times W_r$ that does not depend of the first factor, that is, $H\circ\pi \in C^\infty\left(\mathfrak{m}_r^*\times W_r\right)^{G_m}$. The reconstruction equations show that for any $[g,\rho,w] \in Y_r$,

$$X_H([g,\rho,w]) = T_{(g,\rho,w)}\pi(X_\mathfrak{m}(g,\rho,w), X_{\mathfrak{m}_r^*}(g,\rho,w), X_W(g,\rho,w)),$$

where $X_\mathfrak{m}(g,\rho,w)$, $X_{\mathfrak{m}_r^*}(g,\rho,w)$, and $X_{W_r}(g,\rho,w)$ are determined by the expressions

$$X_\mathfrak{m}(g,\rho,w) = T_e L_g(D_{\mathfrak{m}_r^*}(H\circ\pi)(\rho,w)), \tag{5.81}$$

$$X_{W_r}(g,\rho,w) = \omega_W^\sharp(D_{W_r}(H\circ\pi)(\rho,w)), \tag{5.82}$$

$$X_{\mathfrak{m}_r^*}(g,\rho,w) = \mathbb{P}_{\mathfrak{m}^*}\left(\mathrm{ad}^*_{D_{\mathfrak{m}_r^*}(H\circ\pi)}\rho\right) + \mathrm{ad}^*_{D_{\mathfrak{m}_r^*}(H\circ\pi)}\mathbf{J}_W(w), \tag{5.83}$$

where $\mathbb{P}_{\mathfrak{m}^*} : \mathfrak{g}^* \to \mathfrak{m}^*$ is the projection according to the splitting (5.78) and $\omega_W^\sharp : W^* \to W$ is the isomorphism associated to the symplectic form ω_W in W.

We now assume that X_Q is spectrally unstable, which implies by part **(iv)** that the Hamiltonian vector field $X_{H_S^\xi}$ on the symplectic slice $S_{Y_r} = \{[e, 0, w] \mid w \in W_r\}$ exhibits an unstable equilibrium at $[e, 0, 0]$. Notice now that the Hamiltonian vector field $X_{H_S^\xi}$ is given by the projection onto Y_r via $T\pi$ of the vector field $(0, 0, X_{W_r}^{H_S^\xi})$ in $G \times \mathfrak{m}_r^* \times W_r$ determined by

$$X_{W_r}^{H_S^\xi} = \omega_W^\sharp (D_{W_r} (H \circ \pi)(0, w)) - (\mathbb{P}_{\mathfrak{g}_m}\xi)_W (w) = X_W^H(e, 0, w) - (\mathbb{P}_{\mathfrak{h}}\xi)_W (w),$$

where $\mathbb{P}_{\mathfrak{g}_m} : \mathfrak{g} \to \mathfrak{g}_m$ is the projection according to the splitting (5.77), $(\mathbb{P}_{\mathfrak{g}_m}\xi)_W \in \mathfrak{X}(W_r)$ is the infinitesimal generator associated to $\mathbb{P}_{\mathfrak{g}_m}\xi \in \mathfrak{g}_m$ using the G_m–action on W_r, and X_W^H is the vector field in (5.82) that determines the dynamics induced by H on the space W_r. The instability of $X_{H_S^\xi}$ at $[e, 0, 0]$ implies the same feature for X_W^H at $(e, 0, 0)$ and hence the K–instability of the relative equilibrium $[e, 0, 0]$ of X_H, for any subgroup $K \subset G$. ■

Proof of Proposition 4.13

(i) It is a straightforward consequence of the chain rule as $\mathbf{d}H(g, \mu) = 0$ implies that $\mathbf{d}H^g(e, \mu) = 0$.

(ii) Relation (4.17) is a consequence of the following general fact about Hessians: let $m \in M$ and $n \in N$, with M and N smooth manifolds and let $\psi : M \to N$ be a smooth map such that $\psi(m) = n$. Let $f \in C^\infty(N)$ with $\mathbf{d}f(n) = 0$. Then $\mathbf{d}^2(\psi^* f)(m) = T_m^*\psi(\mathbf{d}^2 f(n))$, that is, for any $v, w \in T_m M$:

$$\mathbf{d}^2(\psi^* f)(m)(v, w) = \mathbf{d}^2 f(n)(T_m\psi \cdot v, T_m\psi \cdot w).$$

In order to establish relation (4.18) note that the map $\Phi_g : \mathfrak{g} \times \mathfrak{g}^* \longrightarrow T_{(g,\mu)} (G \times \mathfrak{g}^*)$ is a symplectomorphism and hence a Poisson map. Expression (4.18) follows from [OR04, Proposition 4.1.19].

(iii) By Lemma 5.2, the Hamiltonian vector field X_{Q^g} is determined by the relation $\mathbf{i}_{X_{Q^g}}\omega(e, \mu) = \mathbf{d}Q^g$ or, more explicitly, by

$$\omega(e, \mu) (X_{Q^g}(\xi, \tau), (\eta, \rho)) = \mathbf{d}Q^g(\xi, \tau) \cdot (\eta, \rho), \quad \text{for any} \quad (\xi, \tau), (\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*.$$

Using the expression of the canonical symplectic form of T^*G in body coordinates (see for instance [OR04, Expression (6.2.5)]), this equality can be rewritten as

$$\langle \rho, \pi_{\mathfrak{g}} (X_{Q^g}(\xi, \tau)) \rangle - \langle \pi_{\mathfrak{g}^*} (X_{Q^g}(\xi, \tau)), \eta \rangle + \langle \mu, \mathrm{ad}_{\pi_{\mathfrak{g}}\left(X_{Q^g}(\xi,\tau)\right)}\eta \rangle = \langle \mathrm{Hess}(\xi, \tau), (\eta, \rho) \rangle. \tag{5.84}$$

Let now $\mathbf{pr}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ and $\mathbf{pr}_{\mathfrak{g}^*} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ be the maps defined by $\mathbf{pr}_{\mathfrak{g}}(\eta, \rho) := (\eta, 0)$ and $\mathbf{pr}_{\mathfrak{g}^*}(\eta, \rho) := (0, \rho)$, for any $(\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*$, and $\mathbf{i}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}^*$ and $\mathbf{i}_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$, the canonical injections. Since (5.84) holds for $(\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*$ arbitrary, we apply it to vectors of the form $\mathbf{pr}_{\mathfrak{g}}(\eta, \rho) = (\eta, 0)$ and $\mathbf{pr}_{\mathfrak{g}^*}(\eta, \rho) = (0, \rho)$ and we obtain the following two equalities

$$\begin{aligned} \langle \mathrm{Hess}(\xi, \tau), \mathbf{pr}_{\mathfrak{g}^*}(\eta, \rho) \rangle &= \langle \pi_{\mathfrak{g}^*}(\eta, \rho), \pi_{\mathfrak{g}} (X_{Q^g}(\xi, \tau)) \rangle, \\ \langle \mathrm{Hess}(\xi, \tau), \mathbf{pr}_{\mathfrak{g}}(\eta, \rho) \rangle &= -\langle \pi_{\mathfrak{g}^*} (X_{Q^g}(\xi, \tau)), \pi_{\mathfrak{g}}(\eta, \rho) \rangle + \langle \mu, \mathrm{ad}_{\pi_{\mathfrak{g}}\left(X_{Q^g}(\xi,\tau)\right)}\pi_{\mathfrak{g}}(\eta, \rho) \rangle. \end{aligned}$$

Since $(\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*$ in these expressions are arbitrary, they can be rewritten as:

$$\mathbf{pr}_{\mathfrak{g}^*}^* \mathrm{Hess}(\xi, \tau) = \pi_{\mathfrak{g}^*}^* (\pi_{\mathfrak{g}} (X_{Q^g}(\xi, \tau))), \tag{5.85}$$

$$\mathbf{pr}_{\mathfrak{g}}^* \mathrm{Hess}(\xi, \tau) = -\pi_{\mathfrak{g}}^* (\pi_{\mathfrak{g}^*} (X_{Q^g}(\xi, \tau))) + \pi_{\mathfrak{g}}^* \left(\mathrm{ad}_{\pi_{\mathfrak{g}}\left(X_{Q^g}(\xi,\tau)\right)}^*(\mu) \right). \tag{5.86}$$

We now apply $\pi_{\mathfrak{g}^*}$ and $\pi_{\mathfrak{g}}$ to both sides of (5.85) and (5.86), respectively, and we notice that $\pi_{\mathfrak{g}^*} \circ \mathbf{pr}^*_{\mathfrak{g}^*} = \pi_{\mathfrak{g}^*}$, $\pi_{\mathfrak{g}} \circ \mathbf{pr}^*_{\mathfrak{g}} = \pi_{\mathfrak{g}}$, $\pi^*_{\mathfrak{g}} = \mathbf{i}_{\mathfrak{g}}$, $\pi^*_{\mathfrak{g}^*} = \mathbf{i}_{\mathfrak{g}^*}$, $\pi_{\mathfrak{g}} \circ \mathbf{i}_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$, and $\pi_{\mathfrak{g}^*} \circ \mathbf{i}_{\mathfrak{g}^*} = \mathrm{id}_{\mathfrak{g}^*}$. We obtain

$$\pi_{\mathfrak{g}}\left(X_{Q^g}(\xi,\tau)\right) = \pi_{\mathfrak{g}^*}\mathrm{Hess}(\xi,\tau), \tag{5.87}$$

$$\pi_{\mathfrak{g}^*}\left(X_{Q^g}(\xi,\tau)\right) = -\pi_{\mathfrak{g}}\left(\mathrm{Hess}(\xi,\tau)\right) + \mathrm{ad}^*_{\pi_{\mathfrak{g}^*}\mathrm{Hess}(\xi,\tau)}\mu, \tag{5.88}$$

which is equivalent to (4.19). ■

5.7 Linear stability and instability of the standard and generalized orbitron relative equilibria

The linearization for the regular branches. The goal in this paragraph is determining the linear Hamiltonian vector fields X_Q in the stability space W used in the proof of Theorem 4.5 by using the expression (4.16) in Proposition 4.12. Notice that this is indeed possible due to the Abelian character of our symmetry group that ensures that in this situation $G_\mu = G$ and hence the coincidence of the tangent spaces $T_m(G_\mu \cdot m)$ and $T_m(G \cdot m)$ that is necessary as a hypothesis in this statement. We start by writing down the decomposition (4.15) that in this case can be achieved by noting that

$$W^\omega = \mathrm{span}\left\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \boldsymbol{\tau}_4\right\} \tag{5.89}$$

with $\boldsymbol{\tau}_1 = (\mathbf{e}_3, r\mathbf{e}_2, \mathbf{0}, \mathbf{0})$ $\boldsymbol{\tau}_2 = (-\mathbf{e}_3, \mathbf{0}, \mathbf{0}, -Mr\xi_1^0\mathbf{e}_1)$, $\boldsymbol{\tau}_3 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_1)$ $\boldsymbol{\tau}_4 = (\mathbf{0}, \mathbf{e}_1, -Mr\xi_1^0\mathbf{e}_3, \mathbf{0})$. If we use as a basis for W the vectors introduced in (5.55) and for W^ω the ones those that we just described, it is easy to see that the matrix expressions of the inclusion $\boldsymbol{i}_W : W \hookrightarrow T_{\mathbf{z}_0}\left(SE(3) \times \mathfrak{se}(3)^*\right) \simeq \mathbb{R}^{12}$ and the projection $\mathbb{P}_W : T_{\mathbf{z}_0}\left(SE(3) \times \mathfrak{se}(3)^*\right) \simeq \mathbb{R}^{12} \longrightarrow W$, where $\mathbb{R}^{12}$ is endowed with the canonical basis, are given by

$$\boldsymbol{i}_W = \left(\mathbf{u}'_1|\mathbf{u}'_2|\mathbf{u}'_3|\mathbf{u}'_4|\mathbf{u}'_5|\mathbf{u}'_6|\mathbf{u}'_7|\mathbf{u}'_8\right), \tag{5.90}$$

$$\mathbb{P}_W = \left(\mathbf{e}'_8|\mathbf{e}'_7|\mathbf{e}'_2|\mathbf{e}'_1| - \frac{1}{r}\mathbf{e}'_2|\mathbf{e}'_6|\mathbf{e}'_3|\mathbf{e}'_4| - \frac{1}{Mr\xi_1^0}\mathbf{e}'_2|\mathbf{0}|\mathbf{e}'_5\right), \tag{5.91}$$

where the apostrophes stand for the transposition operation and the vertical indicate matrix concatenation. The linearization of the Hamiltonian vector field associated to the augmented Hamiltonian at the relative equilibria of the standard orbitron can be immediately obtained by using the expression (4.22) together with the Hessian already computed in (5.50) in the context of the proof of Theorem 4.5. The resulting expression is inserted in (4.16) using the injection (5.90) and the projection (5.91) and yields the following matrix for X_Q:

$$\begin{pmatrix}
0 & -\xi_1^0 r & 0 & 0 & 0 & 0 & 0 & 0 \\
\xi_1^0 r\dfrac{4h^2-r^2}{r^2+h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_1^0 - \dfrac{\Pi_0}{I_1} & 0 & 0 & 0 & -\frac{1}{3}{\xi_1^0}^2 M(r^2+h^2) \\
0 & 0 & -\xi_1^0 + \dfrac{\Pi_0}{I_1} & 0 & 0 & -{\xi_1^0}^2 Mr & -\frac{1}{3}{\xi_1^0}^2 M(r^2+h^2) & 0 \\
0 & 0 & -\dfrac{M\xi_1^0 r}{I_1} & 0 & 0 & {\xi_1^0}^2 M\dfrac{2h^2-3r^2}{r^2+h^2} & -2{\xi_1^0}^2 Mr & 0 \\
0 & 0 & 0 & 0 & \dfrac{1}{M} & 0 & 0 & \xi_1^0 r \\
0 & 0 & 0 & \dfrac{1}{I_1} & 0 & 0 & 0 & -\xi_1^0 \\
0 & 0 & \dfrac{1}{I_1} & 0 & 0 & 0 & \xi_1^0 & 0
\end{pmatrix}, \tag{5.92}$$

where $\Pi_0 = I_3(\xi_1^0 - \xi_2)$. An analog expression can be obtained for the linearization X_Q at the regular relative equilibria of the generalized orbitron:

$$\begin{pmatrix}
0 & -\xi_1^0 r & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4\mu_m}{Mr\xi_1^0}\left(2f_1' + r^2 f_1''\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_1^0 - \frac{\Pi_0}{I_1} & 0 & 0 & 0 & -\mu_m f_0 \\
0 & 0 & -\xi_1^0 + \frac{\Pi_0}{I_1} & 0 & 0 & 2\mu_m r f_1' & -\mu_m f_0 & 0 \\
0 & 0 & -\frac{M\xi_1^0 r}{I_1} & 0 & 0 & \mu_m f_2'' & 4\mu_m r f_1' & 0 \\
0 & 0 & 0 & 0 & \frac{1}{M} & 0 & 0 & \xi_1^0 r \\
0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & -\xi_1^0 \\
0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & \xi_1^0 & 0
\end{pmatrix}.$$

The linearization for the singular branches. The same scheme can be reproduced for the singular branches by using the stability space introduced in (5.57). In this case, it can be shown that $W^\omega = \operatorname{span}\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$, with $\boldsymbol{\tau}_1 = (\mathbf{e}_3, \mathbf{0}, \mathbf{0}, \mathbf{0})$ and $\boldsymbol{\tau}_2 = (\mathbf{0}, \mathbf{0}, \mathbf{e}_3, \mathbf{0})$, which yields the following matrix expressions for the inclusion and the projection:

$$\begin{aligned}
\boldsymbol{i}_W &= (\mathbf{u}_1'|\mathbf{u}_2'|\mathbf{u}_3'|\mathbf{u}_4'|\mathbf{u}_5'|\mathbf{u}_6'|\mathbf{u}_7'|\mathbf{u}_8'|\mathbf{u}_9'|\mathbf{u}_{10}')\,, \\
\mathbb{P}_W &= (\mathbf{e}_{10}'|\mathbf{e}_9'|\mathbf{0}|\mathbf{e}_8'|\mathbf{e}_7'|\mathbf{e}_2'|\mathbf{e}_6'|\mathbf{e}_5'|\mathbf{0}'|\mathbf{e}_4'|\mathbf{e}_3'|\mathbf{e}_1')\,.
\end{aligned}$$

Finally, the matrix expression for X_Q at the singular relative equilibria of the standard orbitron is:

$$\begin{pmatrix}
0 & -3\frac{\mu_0\mu_m q}{\pi h^4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{M} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\xi_1 & 0 & 0 & \frac{3}{2}\frac{\mu_0\mu_m q}{\pi h^4} & 0 & 0 & 0 \\
0 & 0 & \xi_1 & 0 & 0 & 0 & 0 & \frac{3}{2}\frac{\mu_0\mu_m q}{\pi h^4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\xi_1 + \frac{\Pi_0}{I_1} & 0 & 0 & \frac{1}{2}\frac{\mu_0\mu_m q}{\pi h^2} & 0 \\
0 & 0 & 0 & 0 & \xi_1 - \frac{\Pi_0}{I_1} & 0 & 0 & 0 & 0 & \frac{1}{2}\frac{\mu_0\mu_m q}{\pi h^2} \\
0 & 0 & \frac{1}{M} & 0 & 0 & 0 & 0 & -\xi_1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{M} & 0 & 0 & \xi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & 0 & -\xi_1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & \xi_1 & 0
\end{pmatrix}, \tag{5.93}$$

where $\Pi_0 = I_3(\xi_1 - \xi_2)$. An analog expression can be obtained for the linearization X_Q at the singular relative equilibria of the generalized orbitron:

$$
\begin{pmatrix}
0 & \mu_m f_2'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{M} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\xi_1 & 0 & 0 & 2\mu_m f_1' & 0 & 0 & 0 \\
0 & 0 & \xi_1 & 0 & 0 & 0 & 0 & 2\mu_m f_1' & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\xi_1 + \frac{\Pi_0}{I_1} & 0 & 0 & -\mu_m f_0 & 0 \\
0 & 0 & 0 & 0 & \xi_1 - \frac{\Pi_0}{I_1} & 0 & 0 & 0 & 0 & -\mu_m f_0 \\
0 & 0 & \frac{1}{M} & 0 & 0 & 0 & 0 & -\xi_1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{M} & 0 & 0 & \xi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & 0 & 0 & -\xi_1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{I_1} & 0 & 0 & \xi_1 & 0
\end{pmatrix}. \tag{5.94}
$$

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